Online Appendices for "Full Substitutability"

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B Proof of the Main Equivalence Result

In this Appendix, we present a proof of Theorem A.1.

We assume throughout that $\Omega = \Omega_i$ without loss of generality as all of the analysis here considers only the sets of trades demanded by i and, for any price vectors p and $\bar{p}$ such that $p_{\Omega_i} = \bar{p}_{\Omega_i}$, we have that $D_i(p) = D_i(\bar{p})$.

To prove Theorem A.1, we prove seven lemmata; we first show that all three demand language concepts of full substitutability are equivalent.

Lemma 1. *The DFS, DEFS, and DCFS conditions are all equivalent.*

Proof. It is immediate that DEFS and DCFS each imply DFS. To complete the proof, we show that DFS implies DEFS and that DFS implies DCFS.

DFS $\Rightarrow$ DEFS: We first show that Part 1 of DFS implies Part 1 of DEFS. Consider two price vectors p, p' such that $p_\omega = p'_\omega$ for all $\omega \in \Omega_{i\rightarrow}$ and $p_\omega \geq p'_\omega$ for all $\omega \in \Omega_{\rightarrow i}$, and let $\tilde{\Omega} \equiv \{\omega \in \Omega : p_\omega > p'_\omega\}$; note that $\tilde{\Omega} \subseteq \Omega_{\rightarrow i}$. Fix an arbitrary $\Psi \in D_i(p)$; we need to show that there exists a set $\Psi' \in D_i(p')$ that satisfies the requirements of Part 1 of DEFS.

Let q be given by

$$q_\omega = \begin{cases} p_\omega - \varepsilon & \omega \in \Psi_{\rightarrow i} \text{ or } \omega \in [\Omega \smallsetminus \Psi]_{i\rightarrow} \\ p_\omega + \varepsilon & \omega \in [\Omega \smallsetminus \Psi]_{\rightarrow i} \text{ or } \omega \in \Psi_{i\rightarrow} \end{cases}$$

for some sufficiently small $\varepsilon > 0$. Let q' be given by

$$q'_\omega = \begin{cases} p'_\omega & \omega \in \tilde{\Omega} \\ q_\omega & \omega \in \Omega \setminus \tilde{\Omega} \end{cases}$$

$$= \begin{cases} p'_\omega & \omega \in \tilde{\Omega} \\ p_\omega - \varepsilon & \omega \in \Psi_{\to i} \setminus \tilde{\Omega} \text{ or } \omega \in [\Omega \setminus \Psi]_{i\to} \setminus \tilde{\Omega} \\ p_\omega + \varepsilon & \omega \in [\Omega \setminus \Psi]_{\to i} \setminus \tilde{\Omega} \text{ or } \omega \in \Psi_{i\to} \setminus \tilde{\Omega}. \end{cases}$$

$$= \begin{cases} p'_\omega & \omega \in \tilde{\Omega} \\ p'_\omega - \varepsilon & \omega \in \Psi_{\to i} \setminus \tilde{\Omega} \text{ or } \omega \in [\Omega \setminus \Psi]_{i\to} \setminus \tilde{\Omega} \\ p'_\omega + \varepsilon & \omega \in [\Omega \setminus \Psi]_{\to i} \setminus \tilde{\Omega} \text{ or } \omega \in \Psi_{i\to} \setminus \tilde{\Omega}. \end{cases}$$

and let $\Psi' \in D_i(q')$. Let $\bar{q}'$ be given by

$$\bar{q}'_\omega = \begin{cases} q'_\omega - \delta & \omega \in \Psi'_{\to i} \text{ or } \omega \in [\Omega \setminus \Psi']_{i\to} \\ q'_\omega + \delta & \omega \in [\Omega \setminus \Psi']_{\to i} \text{ or } \omega \in \Psi'_{i\to}. \end{cases}$$

$$= \begin{cases} p'_\omega - \delta & \omega \in \Psi'_{\to i} \cap \tilde{\Omega} \text{ or } \omega \in [\Omega \setminus \Psi']_{i\to} \cap \tilde{\Omega} \\ p'_\omega + \delta & \omega \in [\Omega \setminus \Psi']_{\to i} \cap \tilde{\Omega} \text{ or } \omega \in \Psi'_{i\to} \cap \tilde{\Omega} \\ p'_\omega - \varepsilon - \delta & \omega \in [\Psi' \cap \Psi]_{\to i} \setminus \tilde{\Omega} \text{ or } \omega \in [\Omega \setminus (\Psi' \cup \Psi)]_{i\to} \setminus \tilde{\Omega} \\ p'_\omega - \varepsilon + \delta & \omega \in [\Psi \setminus \Psi']_{\to i} \setminus \tilde{\Omega} \text{ or } \omega \in [\Psi' \setminus \Psi]_{i\to} \setminus \tilde{\Omega} \\ p'_\omega + \varepsilon - \delta & \omega \in [\Psi' \setminus \Psi]_{\to i} \setminus \tilde{\Omega} \text{ or } \omega \in [\Psi \setminus \Psi']_{i\to} \setminus \tilde{\Omega} \\ p'_\omega + \varepsilon + \delta & \omega \in [\Omega \setminus (\Psi' \cup \Psi)]_{\to i} \setminus \tilde{\Omega} \text{ or } \omega \in [\Psi' \cap \Psi]_{i\to} \setminus \tilde{\Omega} \end{cases}$$

for some sufficiently small $\delta < \varepsilon$. Finally, let $\bar{q}$ be given by

$$\bar{q}_\omega = \begin{cases} q_\omega & \omega \in \tilde{\Omega} \\ \bar{q}'_\omega & \omega \in \Omega \smallsetminus \tilde{\Omega}. \end{cases}$$

$$= \begin{cases} p_\omega - \varepsilon & \omega \in \Psi_{\to i} \cap \tilde{\Omega} \text{ or } \omega \in [\Omega \smallsetminus \Psi]_{i\to} \cap \tilde{\Omega} \\ p_\omega + \varepsilon & \omega \in [\Omega \smallsetminus \Psi]_{\to i} \cap \tilde{\Omega} \text{ or } \omega \in \Psi_{i\to} \cap \tilde{\Omega} \\ p_\omega - \varepsilon - \delta & \omega \in [\Psi' \cap \Psi]_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in [\Omega \smallsetminus (\Psi' \cup \Psi)]_{i\to} \smallsetminus \tilde{\Omega} \\ p_\omega - \varepsilon + \delta & \omega \in [\Psi \smallsetminus \Psi']_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in [\Psi' \smallsetminus \Psi]_{i\to} \smallsetminus \tilde{\Omega} \\ p_\omega + \varepsilon - \delta & \omega \in [\Psi' \smallsetminus \Psi]_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in [\Psi \smallsetminus \Psi']_{i\to} \smallsetminus \tilde{\Omega} \\ p_\omega + \varepsilon + \delta & \omega \in [\Omega \smallsetminus (\Psi' \cup \Psi)]_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in [\Psi' \cap \Psi]_{i\to} \smallsetminus \tilde{\Omega}. \end{cases}$$

$$= \begin{cases} q_\omega & \omega \in \Psi_{\to i} \cap \tilde{\Omega} \text{ or } \omega \in [\Omega \smallsetminus \Psi]_{i\to} \cap \tilde{\Omega} \\ q_\omega & \omega \in [\Omega \smallsetminus \Psi]_{\to i} \cap \tilde{\Omega} \text{ or } \omega \in \Psi_{i\to} \cap \tilde{\Omega} \\ q_\omega - \delta & \omega \in [\Psi' \cap \Psi]_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in [\Omega \smallsetminus (\Psi' \cup \Psi)]_{i\to} \smallsetminus \tilde{\Omega} \\ q_\omega + \delta & \omega \in [\Psi \smallsetminus \Psi']_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in [\Psi' \smallsetminus \Psi]_{i\to} \smallsetminus \tilde{\Omega} \\ q_\omega - \delta & \omega \in [\Psi' \smallsetminus \Psi]_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in [\Psi \smallsetminus \Psi']_{i\to} \smallsetminus \tilde{\Omega} \\ q_\omega + \delta & \omega \in [\Omega \smallsetminus (\Psi' \cup \Psi)]_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in [\Psi' \cap \Psi]_{i\to} \smallsetminus \tilde{\Omega}. \end{cases}$$

We first show five intermediate results on the effects of our price perturbations.

Fact 1: $D_i(q) = \{\Psi\}$. We have, for any $\Phi \neq \Psi$, that[1]

$$U_i(\Psi; q) - U_i(\Phi; q) = U_i(\Psi; p) - U_i(\Phi; p) + |\Psi \ominus \Phi|\varepsilon \geq |\Psi \ominus \Phi|\varepsilon > 0.$$

where the equality follows from the definition of q, the first inequality follows from the fact that Ψ is optimal at p, i.e., $\Psi \in D_i(p)$, and the last inequality follows as

[1] Here, we use $\ominus$ to denote the symmetric difference between two sets, i.e., $\Psi \ominus \Phi = (\Psi \smallsetminus \Phi) \cup (\Phi \smallsetminus \Psi)$.

$\Phi \neq \Psi$. Thus $D_i(q) = \{\Psi\}$.

Fact 2: $D_i(\bar{q}) = \{\Psi\}$. Consider an arbitrary $\Phi \in D_i(q)$ and an arbitrary $\Xi \notin D_i(q)$. We have that

$$U_i([\Phi; \bar{q}]) - U_i([\Xi; \bar{q}]) \geq U_i([\Phi; q]) - U_i([\Xi; q]) - |\Phi \ominus \Xi|\delta > 0,$$

where the first inequality follows from the definition of $\bar{q}$ and the second inequality follows as Φ is optimal at q, Ξ is not optimal at q, and δ is sufficiently small. Thus, $\Xi \notin D_i(\bar{q})$ and so $D_i(\bar{q}) \subseteq D_i(q)$. Combining this observation with Fact 1 yields $D_i(\bar{q}) = \{\Psi\}$.

Fact 3: $D_i(q') \subseteq D_i(p')$. Consider an arbitrary $\Phi \in D_i(p')$ and an arbitrary $\Xi \notin D_i(p')$. We have that

$$U_i([\Phi; q']) - U_i([\Xi; q']) \geq U_i([\Phi; p']) - U_i([\Xi; p']) - |\Phi \ominus \Xi|\varepsilon > 0,$$

where the first inequality follows from the definition of q' and the second inequality follows as Φ is optimal at p', Ξ is not optimal at p', and ε is sufficiently small. Thus, $\Xi \notin D_i(q')$ and so $D_i(q') \subseteq D_i(p')$.

Fact 4: $D_i(\bar{q}') \subseteq D_i(q')$. Consider an arbitrary $\Phi \in D_i(q')$ and an arbitrary $\Xi \notin D_i(q')$. We have that

$$U_i([\Phi; \bar{q}']) - U_i([\Xi; \bar{q}']) \geq U_i([\Phi; q']) - U_i([\Xi; q']) - |\Phi \ominus \Xi|\delta > 0,$$

where the first inequality follows from the definition of q' and the second inequality follows as Φ is optimal at q', Ξ is not optimal at q', and δ is sufficiently small. Thus, $D_i(\bar{q}') \subseteq D_i(q')$.

Fact 5: $D_i(\bar{q}') = \{\Psi'\}$**.** We have that, for any $\Phi' \neq \Psi'$,

$$U_i(\Psi';\bar{q}') - U_i(\Phi';\bar{q}') = U_i(\Psi';q') - U_i(\Phi;q') + |\Psi' \ominus \Phi'|\delta \geq |\Psi' \ominus \Phi'|\delta > 0.$$

where the equality follows from the definition of $\bar{q}'$, the first inequality follows from the fact that Ψ' is optimal at q', i.e., $\Psi' \in D_i(q')$, and the last inequality follows as $\Phi' \neq \Psi'$. Thus $D_i(\bar{q}') = \{\Psi'\}$.

By Part 1 of DFS, since $D_i(\bar{q}) = \{\Psi\}$ by Fact 2 and $D_i(\bar{q}') = \{\Psi'\}$ by Fact 5, we have that $\{\omega \in \Psi'_{\to i} : p_\omega = p'_\omega\} \subseteq \Psi_{\to i}$ and $\Psi_{i\to} \subseteq \Psi'_{i\to}$. Thus, as $\Psi' \in D_i(p')$ by Facts 3–5, we have that Ψ' satisfies the requirements of Part 1 of DEFS.

The proof that Part 2 of DFS implies Part 2 of DEFS is analogous.

DFS $\Rightarrow$ DCFS: We first show that Part 1 of DFS implies Part 1 of DCFS. Consider two price vectors p, p' such that $p_\omega = p'_\omega$ for all $\omega \in \Omega_{i\to}$ and $p_\omega \geq p'_\omega$ for all $\omega \in \Omega_{\to i}$, and let $\tilde{\Omega} \equiv \{\omega \in \Omega : p_\omega > p'_\omega\}$; note that $\tilde{\Omega} \subseteq \Omega_{\to i}$. Fix an arbitrary $\Psi' \in D_i(p')$; we need to show that there exists a set $\Psi \in D_i(p)$ that satisfies the requirements of Part 1 of DCFS.

Let q' be given by

$$q'_\omega = \begin{cases} p'_\omega - \varepsilon & \omega \in \Psi'_{\to i} \text{ or } \omega \in [\Omega \smallsetminus \Psi']_{i\to} \\ p'_\omega + \varepsilon & \omega \in [\Omega \smallsetminus \Psi']_{\to i} \text{ or } \omega \in \Psi'_{i\to} \end{cases}$$

for some small $\varepsilon > 0$. Let q be given by

$$q_\omega = \begin{cases} p_\omega & \omega \in \tilde{\Omega} \\ q'_\omega & \omega \in \Omega \smallsetminus \tilde{\Omega} \end{cases}$$

$$= \begin{cases} p_\omega & \omega \in \tilde{\Omega} \\ p'_\omega - \varepsilon & \omega \in \Psi'_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in [\Omega \smallsetminus \Psi']_{i\to} \smallsetminus \tilde{\Omega} \\ p'_\omega + \varepsilon & \omega \in [\Omega \smallsetminus \Psi']_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in \Psi'_{i\to} \smallsetminus \tilde{\Omega}. \end{cases}$$

$$= \begin{cases} p_\omega & \omega \in \tilde{\Omega} \\ p_\omega - \varepsilon & \omega \in \Psi'_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in [\Omega \smallsetminus \Psi']_{i\to} \smallsetminus \tilde{\Omega} \\ p_\omega + \varepsilon & \omega \in [\Omega \smallsetminus \Psi']_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in \Psi'_{i\to} \smallsetminus \tilde{\Omega}. \end{cases}$$

and let $\Psi \in D_i(q)$. Let $\bar{q}$ be given by

$$\bar{q}_\omega = \begin{cases} q_\omega - \delta & \omega \in \Psi_{\to i} \text{ or } \omega \in [\Omega \smallsetminus \Psi]_{i\to} \\ q_\omega + \delta & \omega \in [\Omega \smallsetminus \Psi]_{\to i} \text{ or } \omega \in \Psi_{i\to}. \end{cases}$$

$$= \begin{cases} p_\omega - \delta & \omega \in \Psi_{\to i} \cap \tilde{\Omega} \text{ or } \omega \in [\Omega \smallsetminus \Psi]_{i\to} \cap \tilde{\Omega} \\ p_\omega + \delta & \omega \in [\Omega \smallsetminus \Psi]_{\to i} \cap \tilde{\Omega} \text{ or } \omega \in \Psi_{i\to} \cap \tilde{\Omega} \\ p_\omega - \varepsilon - \delta & \omega \in [\Psi' \cap \Psi]_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in [\Omega \smallsetminus (\Psi' \cup \Psi)]_{i\to} \smallsetminus \tilde{\Omega} \\ p_\omega - \varepsilon + \delta & \omega \in [\Psi' \smallsetminus \Psi]_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in [\Psi \smallsetminus \Psi']_{i\to} \smallsetminus \tilde{\Omega} \\ p_\omega + \varepsilon - \delta & \omega \in [\Psi \smallsetminus \Psi']_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in [\Psi' \smallsetminus \Psi]_{i\to} \smallsetminus \tilde{\Omega} \\ p_\omega + \varepsilon + \delta & \omega \in [\Omega \smallsetminus (\Psi' \cup \Psi)]_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in [\Psi' \cap \Psi]_{i\to} \smallsetminus \tilde{\Omega}. \end{cases}$$

Finally, let $\vec{q}'$ be given by

$$\vec{q}'_\omega = \begin{cases} q'_\omega & \omega \in \tilde{\Omega} \\ \bar{q}_\omega & \omega \in \Omega \smallsetminus \tilde{\Omega}. \end{cases}$$

$$= \begin{cases} p'_\omega - \varepsilon & \omega \in \Psi'_{\to i} \cap \tilde{\Omega} \text{ or } \omega \in [\Omega \smallsetminus \Psi']_{i\to} \cap \tilde{\Omega} \\ p'_\omega + \varepsilon & \omega \in [\Omega \smallsetminus \Psi']_{\to i} \cap \tilde{\Omega} \text{ or } \omega \in \Psi'_{i\to} \cap \tilde{\Omega} \\ p'_\omega - \varepsilon - \delta & \omega \in [\Psi' \cap \Psi]_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in [\Omega \smallsetminus (\Psi' \cup \Psi)]_{i\to} \smallsetminus \tilde{\Omega} \\ p'_\omega - \varepsilon + \delta & \omega \in [\Psi' \smallsetminus \Psi]_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in [\Psi \smallsetminus \Psi']_{i\to} \smallsetminus \tilde{\Omega} \\ p'_\omega + \varepsilon - \delta & \omega \in [\Psi \smallsetminus \Psi']_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in [\Psi' \smallsetminus \Psi]_{i\to} \smallsetminus \tilde{\Omega} \\ p'_\omega + \varepsilon + \delta & \omega \in [\Omega \smallsetminus (\Psi' \cup \Psi)]_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in [\Psi' \cap \Psi]_{i\to} \smallsetminus \tilde{\Omega}. \end{cases}$$

$$= \begin{cases} q'_\omega & \omega \in \Psi'_{\to i} \cap \tilde{\Omega} \text{ or } \omega \in [\Omega \smallsetminus \Psi']_{i\to} \cap \tilde{\Omega} \\ q'_\omega & \omega \in [\Omega \smallsetminus \Psi']_{\to i} \cap \tilde{\Omega} \text{ or } \omega \in \Psi'_{i\to} \cap \tilde{\Omega} \\ q'_\omega - \delta & \omega \in [\Psi' \cap \Psi]_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in [\Omega \smallsetminus (\Psi' \cup \Psi)]_{i\to} \smallsetminus \tilde{\Omega} \\ q'_\omega + \delta & \omega \in [\Psi' \smallsetminus \Psi]_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in [\Psi \smallsetminus \Psi']_{i\to} \smallsetminus \tilde{\Omega} \\ q'_\omega - \delta & \omega \in [\Psi \smallsetminus \Psi']_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in [\Psi' \smallsetminus \Psi]_{i\to} \smallsetminus \tilde{\Omega} \\ q'_\omega + \delta & \omega \in [\Omega \smallsetminus (\Psi' \cup \Psi)]_{\to i} \smallsetminus \tilde{\Omega} \text{ or } \omega \in [\Psi' \cap \Psi]_{i\to} \smallsetminus \tilde{\Omega}. \end{cases}$$

We first show five intermediate results on the effects of our price perturbations.

Fact 1: $D_i(q') = \{\Psi'\}$. We have, for any $\Phi' \neq \Psi'$, that

$$U_i(\Psi'; q') - U_i(\Phi'; q') = U_i(\Psi'; p') - U_i(\Phi'; p') + |\Psi' \ominus \Phi'|\varepsilon \geq |\Psi' \ominus \Phi'|\varepsilon > 0.$$

where the equality follows from the definition of q', the first inequality follows from the fact that Ψ' is optimal at p', i.e., $\Psi' \in D_i(p')$, and the last inequality follows as $\Phi' \neq \Psi'$. Thus $D_i(q') = \{\Psi'\}$.

Fact 2: $D_i(\bar{q}') = \{\Psi'\}$**.** Consider an arbitrary $\Phi \in D_i(q')$ and an arbitrary $\Xi \notin D_i(q')$. For δ small enough, we have that,

$$U_i([\Phi;\bar{q}']) - U_i([\Xi;\bar{q}']) \geq U_i([\Phi;q']) - U_i([\Xi;q']) - |\Phi \ominus \Xi|\delta > 0,$$

where the first inequality follows from the definition of $\bar{q}'$ and the second inequality follows as Φ is optimal at q', Ξ is not optimal at q', and δ is sufficiently small. Thus, $\Xi \notin D_i(\bar{q}')$ and so $D_i(\bar{q}') \subseteq D_i(q')$. Combining this observation with Fact 1 yields $D_i(\bar{q}') = \{\Psi'\}$.

Fact 3: $D_i(q) \subseteq D_i(p)$**.** Consider an arbitrary $\Phi \in D_i(p)$ and an arbitrary $\Xi \notin D_i(p)$. We have that

$$U_i([\Phi;q]) - U_i([\Xi;q]) \geq U_i([\Phi;p]) - U_i([\Xi;p]) - |\Phi \ominus \Xi|\varepsilon > 0,$$

where the first inequality follows from the definition of q and the second inequality follows as Φ is optimal at p, Ξ is not optimal at p, and ε is sufficiently small. Thus, $\Xi \notin D_i(q)$ and so $D_i(q) \subseteq D_i(p)$.

Fact 4: $D_i(\bar{q}) \subseteq D_i(q)$**.** Consider an arbitrary $\Phi \in D_i(q)$ and an arbitrary $\Xi \notin D_i(q)$. We have that

$$U_i([\Phi;\bar{q}]) - U_i([\Xi;\bar{q}]) \geq U_i([\Phi;q]) - U_i([\Xi;q]) - |\Phi \ominus \Xi|\delta > 0,$$

where the first inequality follows from the definition of $\bar{q}$ and the second inequality follows as Φ is optimal at q, Ξ is not optimal at q, and ε is sufficiently small. Thus, $D_i(\bar{q}) \subseteq D_i(q)$.

Fact 5: $D_i(\bar{q}) = \{\Psi\}$**.** We have that, for any $\Phi \neq \Psi$,

$$U_i(\Psi;\bar{q}) - U_i(\Phi;\bar{q}) = U_i(\Psi;q) - U_i(\Phi;q) + |\Psi \ominus \Phi|\delta \geq |\Psi \ominus \Phi|\delta > 0.$$

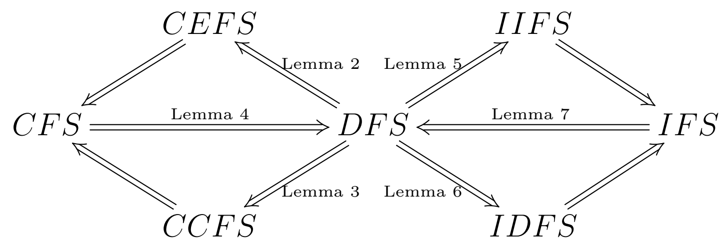

Figure 1: Proof strategy for Theorem A.1. Any unlabeled implication is immediate.

where the equality follows from the definition of $\bar{q}$, the first inequality follows from the fact that Ψ is optimal at q, i.e., $\Psi \in D_i(q)$, and the last inequality follows as $\Phi \neq \Psi$. Thus $D_i(\bar{q}) = \{\Psi\}$.

By Part 1 of DFS, since $\{\Psi'\} = D_i(\bar{q}')$ by Fact 2 and $D_i(\bar{q}) = \{\Psi\}$ by Fact 5, we have that $\{\omega \in \Psi'_{\to i} : p_\omega = p'_\omega\} \subseteq \Psi_{\to i}$ and $\Psi_{i\to} \subseteq \Psi'_{i\to}$. Thus, as $\Psi \in D_i(p)$ by Facts 3–5, we have that Ψ satisfies the requirements of Part 1 of DCFS.

The proof that Part 2 of DFS implies Part 2 of DCFS is analogous.

This completes the proof of Lemma 1. □

We now complete the proof of Theorem A.1 by proving that DFS implies CEFS (Lemma 2), DFS implies CCFS (Lemma 3), DFS implies IIFS (Lemma 5), DFS implies IDFS (Lemma 6), CFS implies DFS (Lemma 4), and IFS implies DFS (Lemma 7), as exemplified in Figure 1.

Lemma 2. *If the preferences of agent i satisfy the DFS condition, then they satisfy the CEFS condition.*

Proof. Consider two finite sets of contracts Y, Z such that $Y_{i\to} = Z_{i\to}$ and $Y_{\to i} \subseteq Z_{\to i}$. Let $Y^* \in C_i(Y)$. We will show that there exists a $Z^* \in C_i(Z)$ such that $(Y_{\to i} \smallsetminus Y^*_{\to i}) \subseteq (Z_{\to i} \smallsetminus Z^*_{\to i})$ and $Y^*_{i\to} \subseteq Z^*_{i\to}$.

Let

$$\tilde{Y} = Y \cup \{(\omega, M) \in X : \omega \in \Omega_{\to i}\} \cup \{(\omega, -M) \in X : \omega \in \Omega_{i\to}\}$$

$$\tilde{Z} = Z \cup \{(\omega, M) \in X : \omega \in \Omega_{\to i}\} \cup \{(\omega, -M) \in X : \omega \in \Omega_{i\to}\}$$

where M is sufficiently large so that i would never choose (ω, M) if $\omega \in \Omega_{\to i}$ or $(\omega, -M)$ if $\omega \in \Omega_{i\to}$.[2] It is immediate that $\tilde{Y}_{i\to} = \tilde{Z}_{i\to}$ and $\tilde{Y}_{\to i} \subseteq \tilde{Z}_{\to i}$. It is also immediate that $C_i(Y) = C_i(\tilde{Y})$ and $C_i(Z) = C_i(\tilde{Z})$.

Let

$$q_\omega^{\tilde{Y}} = \begin{cases} \min\{p_\omega \in \mathbb{R} : \exists(\omega, p_\omega) \in \tilde{Y}\} & \omega \in \Omega_{\to i} \\ \max\{p_\omega \in \mathbb{R} : \exists(\omega, p_\omega) \in \tilde{Y}\} & \omega \in \Omega_{i\to} \end{cases}$$

$$q_\omega^{\tilde{Z}} = \begin{cases} \min\{p_\omega \in \mathbb{R} : \exists(\omega, p_\omega) \in \tilde{Z}\} & \omega \in \Omega_{\to i} \\ \max\{p_\omega \in \mathbb{R} : \exists(\omega, p_\omega) \in \tilde{Z}\} & \omega \in \Omega_{i\to}; \end{cases}$$

note that $q^{\tilde{Y}}$ and $q^{\tilde{Z}}$ are well-defined as, for every $\omega \in \Omega$, there exists a contract $(\omega, p_\omega) \in \tilde{Y} \subseteq \tilde{Z}$ by construction. Moreover, since $\tilde{Y}_{i\to} = \tilde{Z}_{i\to}$ and $\tilde{Y}_{\to i} \subseteq \tilde{Z}_{\to i}$, we have that $q_\omega^{\tilde{Y}} = q_\omega^{\tilde{Z}}$ for all $\omega \in \Omega_{i\to}$ and $q_\omega^{\tilde{Y}} \geq q_\omega^{\tilde{Z}}$ for all $\omega \in \Omega_{\to i}$.

Let $\Psi = \tau(Y^*)$; we have that $\Psi \in D_i(q^{\tilde{Y}})$. Part 1 of DEFS then implies that there exists a $\Psi' \in D_i(q^{\tilde{Z}})$ such that

$$\begin{aligned} \{\omega \in \Psi'_{\to i} : q_\omega^{\tilde{Y}} = q_\omega^{\tilde{Z}}\} &\subseteq \Psi_{\to i} \\ \Psi_{i\to} &\subseteq \Psi'_{i\to}; \end{aligned} \tag{3}$$

let $Z^* = \kappa[\Psi'; q^{\tilde{Z}}]$; note that $Z^* \in C_i(\tilde{Z}) = C_i(Z)$ as Ψ' is optimal at $q^{\tilde{Z}}$ and $q_\omega^{\tilde{Z}}$ is the best price for ω available to i from $\tilde{Z}$. Thus, we can rewrite (3) as

$$\begin{aligned} \{\omega \in \tau(Z^*)_{\to i} : q_\omega^{\tilde{Y}} = q_\omega^{\tilde{Z}}\} &\subseteq \tau(Y^*)_{\to i} \\ \tau(Y^*)_{i\to} &\subseteq \tau(Z^*)_{i\to}; \end{aligned} \tag{4}$$

If $(\omega, p_\omega) \in [Y \smallsetminus Y^*]_{\to i}$, then either:

[2] It is always possible to find M large enough as utility is bounded from above and $u_i(\varnothing) \in \mathbb{R}$.

- $\omega \notin \tau(Y^*) = \Psi$ and so either $\omega \notin \tau(Z^*)$ or $q_\omega^{\tilde{Y}} \neq q_\omega^{\tilde{Z}}$ by (4). In the former case, it is immediate that $(\omega, p_\omega) \notin Z^*_{\to i}$; in the later case, since $q_\omega^{\tilde{Y}} \geq q_\omega^{\tilde{Z}}$, we must have that $q_\omega^{\tilde{Y}} > q_\omega^{\tilde{Z}}$ and so there exists a $(\omega, \bar{p}_\omega) \in Z$ such that $\bar{p}_\omega < p_\omega$, and therefore $(\omega, p_\omega) \notin Z^*_{\to i}$.

- $\omega \in \tau(Y^*)$ but there exists $(\omega, \bar{p}_\omega) \in Y$ such that $\bar{p}_\omega < p_\omega$. In this case, $(\omega, \bar{p}_\omega) \in Z$ as $Y \subseteq Z$, and therefore $(\omega, p_\omega) \notin Z^*_{\to i}$.

Thus, $[Y \smallsetminus Y^*]_{\to i} \subseteq [Z \smallsetminus Z^*]_{\to i}$.

If $(\omega, p_\omega) \in Y^*_{i\to}$, then $\omega \in \tau(Z^*)$ by (4). Moreover, if $(\omega, p_\omega) \in Y^*_{i\to}$ then p_ω is the maximal price in Y for ω and so, as $Y^*_{i\to} = Z^*_{i\to}$, we have that p_ω is the maximal price in Z for ω. Combining these last two observations implies that $(\omega, p_\omega) \in Z^*_{i\to}$, and so $Y^*_{i\to} \subseteq Z^*_{i\to}$.

Thus, Z^* satisfies all the requirements of Part 1 of CEFS.

The proof that DFS implies Part 2 of CEFS is analogous. □

Lemma 3. *If the preferences of agent i satisfy the DFS condition, then they satisfy the CCFS condition.*

Proof. The proof proceeds analogously to the proof of Lemma 2. □

Lemma 4. *If the preferences of agent i satisfy the CFS condition, then they satisfy the DFS condition.*

Proof. We first show that Part 1 of CFS implies Part 1 of DFS. For any agent i and price vector $p \in \mathbb{R}^\Omega$, let

$$X_i(p) \equiv \{(\omega, \hat{p}_\omega) : \omega \in \Omega_{\to i} \text{ and } \hat{p}_\omega \geq p_\omega\} \cup \{(\omega, \hat{p}_\omega) : \omega \in \Omega_{i\to} \text{ and } \hat{p}_\omega \leq p_\omega\};$$

that is, $X_i(p)$ effectively denotes the set of contracts available to agent i under prices p.[3]

[3]By this, we mean that, in principle, an agent pay more for an upstream trade and receive less for a downstream trade.

Let the price vectors $p, p' \in \mathbb{R}^\Omega$ be such that $|D_i(p)| = |D_i(p')| = 1$, $p_\omega = p'_\omega$ for all $\omega \in \Omega_{i\to}$, and $p'_\omega \le p_\omega$ for all $\omega \in \Omega_{\to i}$; let $\{\Psi\} = D_i(p)$ and $\{\Psi'\} = D_i(p')$. Let $Y = X_i(p)$ and $Z = X_i(p')$. Clearly, $Y_{i\to} = Z_{i\to}$ and $Y_{\to i} \subseteq Z_{\to i}$. Furthermore, it is immediate that $\{\kappa([\Psi;p])\} = C_i(Y)$, and similarly, $\{\kappa([\Psi;p'])\} = C_i(Z)$. Thus, Part 1 of CFS implies that

$$Y_{\to i} \smallsetminus [\kappa([\Psi;p])]_{\to i} \subseteq Z_{\to i} \smallsetminus [\kappa([\Psi';p'])]_{\to i} \tag{5}$$

$$[\kappa([\Psi;p])]_{i\to} \subseteq [\kappa([\Psi';p'])]_{i\to}. \tag{6}$$

From (5), we see that, if $\omega \in \tau([\kappa([\Psi';p'])]_{\to i})$, i.e., if $\omega \in \Psi'_{\to i}$, and $p'_\omega = p_\omega$, then $(\omega, p'_\omega) \in [\kappa([\Psi;p])]_{\to i}$, and so $\omega \in \Psi_{\to i}$—in other words, $\{\omega \in \Psi'_{\to i} : p'_\omega = p_\omega\} \subseteq \Psi_{\to i}$. Furthermore, as $[\kappa([\Psi;p])]_{i\to} \subseteq [\kappa([\Psi';p'])]_{i\to}$ by (6) and $p_\omega = p'_\omega$ for each $\omega \in \Omega_{i\to}$, we have that $\Psi'_{i\to} \subseteq \Psi_{i\to}$. Thus, Ψ' satisfies the requirements of Part 1 of DFS.

The proof that Part 2 of CFS implies Part 2 of DFS is analogous. □

Lemma 5. *If the preferences of agent i satisfy the DFS condition, then they satisfy the IIFS condition.*

Proof. It is enough to show that DEFS and DCFS jointly imply IIFS, as DFS implies both DEFS and DCFS by Lemma 1. Take two price vectors $p, p' \in \mathbb{R}^\Omega$ such that $p \le p'$, and let $\Psi \in D_i(p)$ be arbitrary. We will show that there exists a set of trades $\Psi' \in D_i(p')$ such that $e_{i,\omega}(\Psi) \le e_{i,\omega}(\Psi')$ for all $\omega \in \Omega_i$ such that $p_\omega = p'_\omega$.

We let

$$p^\star_\omega = \begin{cases} p'_\omega & \omega \in \Omega_{\to i} \\ p_\omega & \omega \in \Omega_{i\to}; \end{cases}$$

thus, $p^\star_\omega = p_\omega$ for all $\omega \in \Omega_{i\to}$ and $p^\star_\omega \ge p_\omega$ for all $\omega \in \Omega_{\to i}$. Part 1 of DCFS then implies that there exists a $\Psi^\star \in D_i(p^\star)$ such that

$$\{\omega \in \Psi_{\to i} : p_\omega = p^\star_\omega\} \subseteq \Psi^\star_{\to i} \tag{7}$$
$$\Psi^\star_{i\to} \subseteq \Psi_{i\to}.$$

Now, note that $p^\star_\omega = p'_\omega$ for all $\omega \in \Omega_{\to i}$ and $p^\star_\omega \leq p'_\omega$ for all $\omega \in \Omega_{i\to}$. Part 2 of DEFS then implies that there exists a $\Psi' \in D_i(p')$ such that

$$\{\omega \in \Psi'_{i\to} : p^\star_\omega = p'_\omega\} \subseteq \Psi^\star_{i\to} \tag{8}$$
$$\Psi^\star_{\to i} \subseteq \Psi'_{\to i}.$$

Combining (7) and (8) yields

$$\{\omega \in \Psi_{\to i} : p_\omega = p^\star_\omega\} \subseteq \Psi^\star_{\to i} \subseteq \Psi'_{\to i}$$
$$\{\omega \in \Psi'_{i\to} : p^\star_\omega = p'_\omega\} \subseteq \Psi^\star_{i\to} \subseteq \Psi_{i\to}.$$

Recalling the definition of $p^\star$, we obtain

$$\{\omega \in \Psi_{\to i} : p_\omega = p'_\omega\} \subseteq \Psi'_{\to i}$$
$$\{\omega \in \Psi'_{i\to} : p_\omega = p'_\omega\} \subseteq \Psi_{i\to};$$

this implies $e_{i,\omega}(\Psi) \leq e_{i,\omega}(\Psi')$ for all $\omega \in \Omega_i$ such that $p_\omega = p'_\omega$. □

Lemma 6. *If the preferences of agent i satisfy the DFS condition, then they satisfy the ICFS condition.*

Proof. It is enough to show that DEFS and DCFS jointly imply IDFS, as DFS implies both DEFS and DCFS by Lemma 1. Take two price vectors $p, p' \in \mathbb{R}^\Omega$ such that $p \leq p'$, and let $\Psi' \in D_i(p')$ be arbitrary. We will show that there exists a set of trades $\Psi \in D_i(p)$ such that $e_{i,\omega}(\Psi) \leq e_{i,\omega}(\Psi')$ for all $\omega \in \Omega_i$ such that $p_\omega = p'_\omega$.

Let

$$p^\star_\omega = \begin{cases} p'_\omega & \omega \in \Omega_{\to i} \\ \\ p_\omega & \omega \in \Omega_{i\to}; \end{cases}$$

thus, $p^\star_\omega = p'_\omega$ for all $\omega \in \Omega_{\to i}$ and $p^\star_\omega \le p'_\omega$ for all $\omega \in \Omega_{i\to}$. Part 2 of DCFS then implies that there exists a $\Psi^\star \in D_i(p^\star)$ such that

$$\begin{aligned} \{\omega \in \Psi'_{i\to} : p^\star_\omega = p'_\omega\} &\subseteq \Psi^\star_{i\to} \\ \Psi^\star_{\to i} &\subseteq \Psi'_{\to i}. \end{aligned} \tag{9}$$

Now, note that $p^\star_\omega = p_\omega$ for all $\omega \in \Omega_{i\to}$ and $p^\star_\omega \ge p_\omega$ for all $\omega \in \Omega_{\to i}$. Part 1 of DEFS then implies that there exists a $\Psi \in D_i(p)$ such that

$$\begin{aligned} \{\omega \in \Psi_{\to i} : p^\star_\omega = p_\omega\} &\subseteq \Psi^\star_{\to i} \\ \Psi^\star_{i\to} &\subseteq \Psi_{i\to}. \end{aligned} \tag{10}$$

Combining (9) and (10) yields

$$\begin{aligned} \{\omega \in \Psi'_{i\to} : p^\star_\omega = p'_\omega\} \subseteq \Psi^\star_{i\to} \subseteq \Psi_{i\to} \\ \{\omega \in \Psi_{\to i} : p^\star_\omega = p_\omega\} \subseteq \Psi^\star_{\to i} \subseteq \Psi'_{\to i}. \end{aligned}$$

Recalling the definition of $p^\star$, we obtain

$$\begin{aligned} \{\omega \in \Psi'_{i\to} : p_\omega = p'_\omega\} \subseteq \Psi_{i\to} \\ \{\omega \in \Psi_{\to i} : p_\omega = p_\omega\} \subseteq \Psi'_{\to i}; \end{aligned}$$

this implies $e_{i,\omega}(\Psi) \le e_{i,\omega}(\Psi')$ for all $\omega \in \Omega_i$ such that $p_\omega = p'_\omega$. □

Lemma 7. *If the preferences of agent i satisfy the IFS condition, then they satisfy the DFS condition.*

Proof. Let the price vectors $p, p' \in \mathbb{R}^{\Omega}$ be such that $|D_i(p)| = |D_i(p')| = 1$, $p_\omega = p'_\omega$ for all $\omega \in \Omega_{i\to}$, and $p'_\omega \leq p_\omega$ for all $\omega \in \Omega_{\to i}$; let $\{\Psi\} = D_i(p)$ and $\{\Psi'\} = D_i(p')$. As the preferences of i satisfy the IFS condition, we have that $e_{i,\omega}(\Psi') \leq e_{i,\omega}(\Psi)$ for all $\omega \in \Omega_{\to i}$ such that $p_\omega = p'_\omega$. Thus, if $p_\omega = p'_\omega$ and $\omega \in \Psi'$ then $\omega \in \Psi$ and so we have that $\{\omega \in \Psi'_{\to i} : p'_\omega = p_\omega\} \subseteq \Psi_{\to i}$. Moreover, as the preferences of i satisfy the IFS condition, we have that $e_{i,\omega}(\Psi') \leq e_{i,\omega}(\Psi)$ for all $\omega \in \Omega_{i\to}$ such that $p_\omega = p'_\omega$. Thus, if $p_\omega = p'_\omega$ and $\omega \in \Psi$ then $\omega \in \Psi'$ and so, as $p_\omega = p'_\omega$ for all $p_\omega = p'_\omega$ for all $\omega \in \Omega_{i\to}$, we have that $\Psi_{\to i} \subseteq \Psi'_{\to i}$.

The proof that Part 2 of IFS implies Part 2 of DFS is analogous. □

C Proofs of the Results Presented in Sections 4–7

Proof of Proposition 1

Consider the intermediary i. Let Φ (with a typical element φ) denote the set of potential inputs this intermediary faces, and let Ψ (with a typical element ψ) denote the set of potential requests. The cost of using input φ to satisfy request ψ is given by $c_{\varphi,\psi}$. For convenience, when φ and ψ are incompatible, we simply say that $c_{\varphi,\psi} = +\infty$.

Let us now construct a "synthetic" agent $\hat{\imath}$ whose preferences will be identical to those of agent i, yet will be represented in the form of "intermediary with production capacity" preferences as defined in Section 4.2. The full substitutability of the preferences of intermediary i will then follow immediately from Proposition 2.

Agent $\hat{\imath}$ faces the same sets of inputs, Φ, and requests, Ψ, as agent i. Agent $\hat{\imath}$ also has $|\Phi| \times |\Psi|$ machines, indexed by pairs of inputs and requests: machine $m_{\varphi,\psi}$ "corresponds" to an input–request pair (φ, ψ). The costs of intermediary $\hat{\imath}$ are as follows (to avoid confusion, we will denote various costs of agent $\hat{\imath}$ by "$\hat{c}$" with various subindices, while the costs of agent i are denoted by "c" with various subindices):

- For input φ and machine $m_{\varphi,\psi}$ "corresponding" to input φ and some request ψ, the cost $\hat{c}_{\varphi,m_{\varphi,\psi}}$ of using input φ in machine $m_{\varphi,\psi}$ is equal to $c_{\varphi,\psi}$—the cost of using input φ to satisfy request ψ under the original cost structure of agent i.

- For any input $\varphi' \neq \varphi$ and any request ψ, the cost $\hat{c}_{\varphi',m_{\varphi,\psi}}$ is equal to $+\infty$.

- For request ψ and any machine $m_{\varphi,\psi}$ "corresponding" to request ψ and some input φ, the cost $\hat{c}_{m_{\varphi,\psi},\psi}$ of using machine $m_{\varphi,\psi}$ to satisfy request ψ is equal to 0.

- For any request $\psi' \neq \psi$ and any machine $m_{\varphi,\psi}$, the cost $\hat{c}_{m_{\varphi,\psi},\psi'}$ is equal to $+\infty$.

With this construction, the preferences of agents i and $\hat{\imath}$ over sets of inputs and requests are identical. Moreover, the preferences of agent $\hat{\imath}$ are those of "intermediary with production capacity" and are thus fully substitutable (by Proposition 2). Therefore, the "intermediary" preferences of agent i are also fully substitutable.

Proof of Proposition 2

Consider first an "intermediary with production capacity" who has exactly one machine at his disposal. It is immediate that the preferences of such an intermediary are fully substitutable.

Next, consider a general "intermediary with production capacity", i, who has a set of machines M (with a typical element m) at his disposal and faces the set of inputs Φ (with a typical element φ) and the set of potential requests Ψ (with a typical element ψ), with costs as described in Section 4.2. We will show that the preferences of intermediary i can be represented as a "merger" of several (specifically, $|M| + |\Phi| + |\Psi|$) agents with fully substitutable preferences, which by Theorem 4 will imply that the preferences of intermediary i are fully substitutable.

Specifically, consider the following set of artificial agents. First, there are $|\Phi|$ "input dummies", with a typical element $\hat{\varphi}$ for a dummy that corresponds to input φ. Second, there are $|M|$ "machine dummies", with a typical element $\hat{m}$ for a dummy that corresponds to machine m. Finally, there are $|\Psi|$ "request dummies", with a typical element $\hat{\psi}$ for a dummy that corresponds to request ψ.

Each input dummy $\hat{\varphi}$ can only buy one trade: input φ. He can also form $|M|$ trades as a seller: one trade with every machine dummy $\hat{m}$. We denote the trade between an input dummy $\hat{\varphi}$ (as the seller) and a machine dummy $\hat{m}$ (as the buyer) by $\omega_{\varphi,m}$. Likewise, each request dummy $\hat{\psi}$ can only sell one trade: request ψ. He can also form $|M|$ trades as a buyer: one trade with every machine dummy $\hat{m}$. We denote the trade between a machine dummy $\hat{m}$ (as the seller) and a request dummy $\hat{\psi}$ (as the buyer) by $\omega_{m,\psi}$. Each machine dummy can thus form $|\Phi|$ trades as the buyer (one with each input dummy) and $|\Psi|$ trades as the seller (one with each request dummy).

The preferences of the agents are as follows. Each input dummy and each request dummy has valuation 0 if the number of trades he forms as the seller is equal to the number of trades he forms as the buyer (this number can thus be equal to either 0 or 1), and $-\infty$ if these numbers are not equal. It is immediate that the preferences of input and request dummies are fully substitutable.

The preferences of each machine dummy $\hat{m}$ are as follows. If it buys no trades and sells no trades, its valuation is 0. If it buys exactly one trade, say $\omega_{\varphi,m}$ for some φ, and sells exactly one trade, say $\omega_{m,\psi}$ for some ψ, then its valuation is $-(c_{\varphi,m} + c_{m,\psi})$—the total cost of preparing input φ for request ψ using machine m in the original construction of the utility function of agent i. In all other cases (i.e., when the machine dummy buys or sells more than two trades, or when the number of trades it buys is not equal to the number of trades it sells), the valuation of the machine dummy is $-\infty$. Note that the preferences of the machine dummy are also fully substitutable.

Consider now the "synthetic" agent $\hat{\imath}$ obtained as the merger of the $|\Phi|$ input dummies, $|M|$ machine dummies, and $|\Psi|$ request dummies (see Section 5.2 for the details of the "merger" operation). By Theorem 4, the preferences of agent $\hat{\imath}$ are fully substitutable. Moreover, the valuation of agent $\hat{\imath}$ over any bundle of inputs and requests is identical to the valuation of agent i over that bundle. Thus, the preferences of agent i are fully substitutable.

Proof of Theorem 2

The indirect utility function for $\hat{u}_i^{(\Phi,p_\Phi)}$ is given by

$$\begin{aligned}
\hat{V}_i^{(\Phi,p_\Phi)}(p_{\Omega\smallsetminus\Phi}) &\equiv \max_{\Psi\subseteq\Omega\smallsetminus\Phi}\left\{\max_{\Xi\subseteq\Phi}\left\{u_i(\Psi\cup\Xi)+\sum_{\xi\in\Xi_{\to i}}p_\xi-\sum_{\xi\in\Xi_{\to i}}p_\xi\right\}+\sum_{\psi\in\Psi_{\to i}}p_\xi-\sum_{\psi\in\Psi_{\to i}}p_\xi\right\}\\
&=\max_{\Psi\subseteq\Omega\smallsetminus\Phi}\left\{\max_{\Xi\subseteq\Phi}\left\{u_i(\Psi\cup\Xi)+\sum_{\lambda\in\Xi_{\to i}\cup\Psi_{\to i}}p_\lambda-\sum_{\lambda\in\Xi_{i\to}\cup\Psi_{i\to}}p_\lambda\right\}\right\}\\
&=\max_{\Lambda\subseteq\Omega}\left\{u_i(\Lambda)+\sum_{\lambda\in\Lambda_{\to i}}p_\lambda-\sum_{\lambda\in\Lambda_{i\to}}p_\lambda\right\}.
\end{aligned}$$

Hence, $\hat{V}_i^{(\Phi,p_\Phi)}(p_{\Omega\smallsetminus\Phi})=V_i(p_{\Omega\smallsetminus\Phi},p_\Phi)$. Now, $V_i(p)$ is submodular over $\mathbb{R}^\Omega$ by Theorem 6. As a submodular function restricted to a subspace of its domain is still submodular, $\hat{V}_i^{(\Phi,p_\Phi)}(p_{\Omega\smallsetminus\Phi})$ is submodular over $\mathbb{R}^{\Omega\smallsetminus\Phi}$. Hence, by Theorem 6, we see that $\hat{u}_i^{(\Phi,p_\Phi)}$ is fully substitutable.

Proof of Theorem 3

Fix a set of trades $\Phi\subseteq\Omega_i$ such that $u_i(\Phi)\neq-\infty$ and a vector of prices $\overline{p}_\Phi$ for trades in Φ. Let $\tilde{D}_i$ be the demand function for trades in $\Omega\smallsetminus\Phi$ induced by $\tilde{u}_i^{\Phi,\overline{p}_\Phi}$. Fix two price vectors $p\in\mathbb{R}^{\Omega\smallsetminus\Phi}$ and $p'\in\mathbb{R}^{\Omega\smallsetminus\Phi}$ such that $|\tilde{D}_i(p)|=|\tilde{D}_i(p')|=1$, $p_\omega=p'_\omega$ for all $\omega\in\Omega_{i\to}\smallsetminus\Phi$, and $p_\omega\geq p'_\omega$ for all $\omega\in\Omega_{\to i}\smallsetminus\Phi$. Let $\Psi\in\tilde{D}_i(p)$ be the unique demanded set from $\Omega_i\smallsetminus\Phi$ at price vector p and $\Psi'\in\tilde{D}_i(p')$ be the unique demanded set from $\Omega_i\smallsetminus\Phi$ at price vector p'. Note that since $u_i(\Phi)\neq-\infty$, there exists a vector of prices p^*_Φ for trades in Φ such that, for all $\Xi\in D_i((p,p^*_\Phi))\cup D_i((p',p^*_\Phi))$, we have $\Phi\subseteq\Xi$. Fix an arbitrary $\Xi\in D_i((p,p^*_\Phi))$ and let $\tilde{\Psi}\equiv\Xi\smallsetminus\Phi$.

Claim 1. *We must have* $\tilde{\Psi}=\Psi$.

Proof. Suppose the contrary. Since $\tilde{\Psi} \cup \Phi = \Xi \in D_i((p, p^*_\Phi))$, we must have

$$\begin{aligned} u_i(\Xi) &= u_i(\tilde{\Psi} \cup \Phi) + \sum_{\psi \in \tilde{\Psi}_{i\to}} p_\psi - \sum_{\psi \in \tilde{\Psi}_{\to i}} p_\psi + \sum_{\varphi \in \Phi_{i\to}} p^*_\varphi - \sum_{\varphi \in \Phi_{\to i}} p^*_\varphi \\ &\geq u_i(\Psi \cup \Phi) + \sum_{\psi \in \Psi_{i\to}} p_\psi - \sum_{\psi \in \Psi_{\to i}} p_\psi + \sum_{\varphi \in \Phi_{i\to}} p^*_\varphi - \sum_{\varphi \in \Phi_{\to i}} p^*_\varphi. \end{aligned} \tag{11}$$

The inequality (11) is equivalent to

$$\begin{aligned} u_i(\tilde{\Psi} \cup \Phi) &+ \sum_{\psi \in \tilde{\Psi}_{i\to}} p_\psi - \sum_{\psi \in \tilde{\Psi}_{\to i}} p_\psi + \sum_{\varphi \in \Phi_{i\to}} \overline{p}_\varphi - \sum_{\varphi \in \Phi_{\to i}} \overline{p}_\varphi \\ &\geq u_i(\Psi \cup \Phi) + \sum_{\psi \in \Psi_{i\to}} p_\psi - \sum_{\psi \in \Psi_{\to i}} p_\psi + \sum_{\varphi \in \Phi_{i\to}} \overline{p}_\varphi - \sum_{\varphi \in \Phi_{\to i}} \overline{p}_\varphi. \end{aligned} \tag{12}$$

However, the inequality (12) implies that $\tilde{\Psi} \in \tilde{D}_i(p)$; this contradicts the assumption that $\tilde{D}_i(p) = \{\Psi\}$ given that $\tilde{\Psi} \neq \Psi$. □

The preceding claim implies that we must have $D_i((p, p^*_\Phi)) = \{\Xi\} = \{\tilde{\Psi} \cup \Phi\} = \{\Psi \cup \Phi\}$. A similar argument shows that $D_i((p', p^*_\Phi)) = \{\Psi' \cup \Phi\}$. The full substitutability of u_i then implies that $\{\psi \in \Psi'_{\to i} : p_\psi = p'_\psi\} \subseteq \Psi_{\to i}$ and $\Psi_{i\to} \subseteq \Psi'_{i\to}$.

Proof of Theorem 4

We suppose, by way of contradiction, that u_J does not induce fully substitutable preferences over trades in $\Omega \smallsetminus \Omega^J$. By Corollary 1 of Hatfield et al. (2013), there exist fully substitutable preferences $\tilde{u}_i$ for the agents $i \in I \smallsetminus J$ such that no competitive equilibrium exists for the *modified economy* with

1. set of agents $(I \smallsetminus J) \cup \{J\}$,

2. set of trades $\Omega \smallsetminus \Omega^J$,

3. and valuation function for agent J given by u_J.[4]

[4]Technically, in order to apply Corollary 1 of Hatfield et al. (2013), we must have that for every pair (i, j)

Now, we consider the *original economy* with

1. set of agents I,

2. set of trades Ω,

3. valuations for $i \in I \smallsetminus J$ given by $\tilde{u}_i$, and

4. valuations for $j \in J$ given by u_j.

Let $[\Psi; p]$ be a competitive equilibrium of this economy; such an equilibrium must exist by Theorem 1 of Hatfield et al. (2013).

Claim 2. *$[\Psi \smallsetminus \Omega^J; p_{\Omega \smallsetminus \Omega^J}]$ is a competitive equilibrium of the modified economy.*

Proof. It is immediate that $[\Psi \smallsetminus \Omega^J]_i \in D_i(p_{\Omega \smallsetminus \Omega^J})$ for all $i \in I \smallsetminus J$. Moreover, since Ψ is individually-optimal for each $j \in J$ in the original economy (at prices p),

$$u_j(\Psi) + \sum_{\psi \in \Psi_{j \to}} p_\psi - \sum_{\psi \in \Psi_{\to j}} p_\psi \geq u_j(\Phi) + \sum_{\varphi \in \Phi_{j \to}} p_\varphi - \sum_{\varphi \in \Phi_{\to j}} p_\varphi \tag{13}$$

for every $\Phi \subseteq \Omega$. Summing (13) over all $j \in J$ and simplifying, we obtain

$$\sum_{j \in J} \left(u_j(\Psi) + \sum_{\psi \in \Psi_{j \to}} p_\psi - \sum_{\psi \in \Psi_{\to j}} p_\psi \right) \geq \sum_{j \in J} \left(u_j(\Phi) + \sum_{\varphi \in \Phi_{j \to}} p_\varphi - \sum_{\varphi \in \Phi_{\to j}} p_\psi \right)$$

$$\sum_{j \in J} \left(u_j(\Psi) + \sum_{\psi \in [\Psi \smallsetminus \Omega^J]_{j \to}} p_\psi - \sum_{\psi \in [\Psi \smallsetminus \Omega^J]_{\to j}} p_\psi \right) \geq \sum_{j \in J} \left(u_j(\Phi) + \sum_{\varphi \in [\Phi \smallsetminus \Omega^J]_{j \to}} p_\varphi - \sum_{\varphi \in [\Phi \smallsetminus \Omega^J]_{\to j}} p_\psi \right)$$

$$\sum_{j \in J} u_j(\Psi) + \sum_{\psi \in [\Psi \smallsetminus \Omega^J]_{J \to}} p_\psi - \sum_{\psi \in [\Psi \smallsetminus \Omega^J]_{\to J}} p_\psi \geq \sum_{j \in J} u_j(\Phi) + \sum_{\varphi \in [\Phi \smallsetminus \Omega^J]_{J \to}} p_\varphi - \sum_{\varphi \in [\Phi \smallsetminus \Omega^J]_{\to J}} p_\psi. \quad \square$$

The preceding claim shows that $[\Psi \smallsetminus \Omega^J; p_{\Omega \smallsetminus \Omega^J}]$ is a competitive equilibrium of the modified economy, contradicting the earlier conclusion that no competitive equilibrium exists in the modified economy. Hence, we see that u_J must be fully substitutable.

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of distinct agents in I, there exists a trade ω such that $b(\omega) = i$ and $s(\omega) = j$. For any pair (i, j) of distinct agents in I such that no such trade ω exists, we can augment the economy by adding the requisite trade ω and, if $i \in J$, letting $\bar{u}_i(\Psi \cup \{\omega\}) = u^i(\Psi)$ (and similarly for j). It is immediate that $\bar{u}_i$ is substitutable if and only if u_i is substitutable.

Proof of Theorem 5

The proof of this result is very close to Step 1 of the proof of Theorem 1 of Hatfield et al. (2013). The only differences are that in the Hatfield et al. (2013) results, all trades could be bought out, and the price for buying them out was set to a single large number that was the same for all trades. By contrast, in Theorem 5 of the current paper we allow for the possibility that only a subset of trades can be bought out, and that the prices at which these trades can be bought out can be different, and need not be large. Adapting Step 1 of the proof of Theorem 1 of Hatfield et al. (2013) to the current more general setting is straightforward, but we include the proof for completeness.

Consider the fully substitutable valuation function u_i, and take any trade $\varphi \in \Omega_{i\to} \cap \Phi$. Consider a modified valuation function u_i^φ:

$$u_i^\varphi(\Psi) = \max\{u_i(\Psi), u_i(\Psi \smallsetminus \{\varphi\}) - \Pi_\varphi\}.$$

That is, the valuation $u_i^\varphi(\Psi)$ allows (but does not require) agent i to pay Π_φ instead of executing one particular trade, φ.

Claim 3. *The valuation function u_i^φ is fully substitutable.*

Proof. We consider utility U_i^φ and demand D_i^φ corresponding to valuation u_i^φ. We show that D_i^φ satisfies the IFS condition (Definition 3). Fix two price vectors p and p' such that $p \leq p'$ and $|D_i^\varphi(p)| = |D_i^\varphi(p')| = 1$. Take the unique $\Psi \in D_i^\varphi(p)$ and $\Psi' \in D_i^\varphi(p')$. We need to show that

$$e_{i,\omega}(\Psi) \leq e_{i,\omega}(\Psi') \text{ for all } \omega \in \Omega_i \text{ such that } p_\omega = p'_\omega. \quad (14)$$

Let price vector q coincide with p on all trades other than φ, and set $q_\varphi = \min\{p_\varphi, \Pi_\varphi\}$. Note that if $p_\varphi < \Pi_\varphi$, then $p = q$ and $D_i^\varphi(p) = D_i(p)$. If $p_\varphi > \Pi_\varphi$, then under utility U_i^φ, agent i always wants to execute trade φ at price p_φ, and the only decision is whether to "buy it out" or not at the cost Π_φ; i.e., the agent's effective demand is the same as under price vector q. Thus, $D_i^{\varphi}(p) = \{\Xi \cup \{\varphi\} : \Xi \in D_i(q)\}$. Finally, if $p_\varphi = \Pi_\varphi$, then $p = q$ and $D_i^{\varphi}(p) = D_i(p) \cup \{\Xi \cup \{\varphi\} : \Xi \in D_i(p)\}$. We construct price vector q' corresponding to p' analogously.

Now, if $p_\varphi \leq p'_\varphi < \Pi_\varphi$, then $D_i^{\varphi}(p) = D_i(p)$, $D_i^{\varphi}(p') = D_i(p')$, and thus $e_{i,\omega}(\Psi) \leq e_{i,\omega}(\Psi')$ follows directly from IFS for demand D_i.

If $\Pi_\varphi \leq p_\varphi \leq p'_\varphi$, then (since we assumed that D_i^{φ} was single-valued at p and p') it has to be the case that D_i is single-valued at the corresponding price vectors q and q'. Let $\Xi \in D_i(q)$ and $\Xi' \in D_i(q')$. Then $\Psi = \Xi \cup \{\varphi\}$, $\Psi' = \Xi' \cup \{\varphi\}$, and statement (14) follows from the IFS condition for demand D_i, because $q \leq q'$.

Finally, if $p_\varphi < \Pi_\varphi \leq p'_\varphi$, then $p = q$, Ψ is the unique element in $D_i(p)$, and Ψ' is equal to $\Xi' \cup \{\varphi\}$, where Ξ' is the unique element in $D_i(q')$. Then for $\omega \neq \varphi$, statement (14) follows from IFS for demand D_i, because $p \leq q'$. For $\omega = \varphi$, statement (14) does not need to be checked, because $p_\varphi < p'_\varphi$.

Thus, when $\varphi \in \Omega_{i\to}$, the valuation function u_i^{φ} is fully substitutable. The proof for the case when $\varphi \in \Omega_{\to i}$ is completely analogous. □

To complete the proof of Theorem 5, it is now enough to note that valuation function $\hat{u}_i(\Psi) = \max_{\Xi \subseteq \Psi \cap \Phi} \Big\{ u_i(\Psi \smallsetminus \Xi) - \sum_{\varphi \in \Xi} \Pi_\varphi \Big\}$ can be obtained from the original valuation u_i by allowing agent i to "buy out" all of the trades in set Φ, one by one, and since the preceding claim shows that each such transformation preserves substitutability (and Ω_i is finite), the valuation function $\hat{u}_i$ is substitutable as well.

Proof of Theorem 6

We first show that if the preferences of an agent i are fully substitutable, then those preferences induce a submodular indirect utility function. It is enough to show that for any two trades

$\varphi, \psi \in \Omega_i$ and any prices $p \in \mathbb{R}^\Omega$, $p_\varphi^{\text{high}} > p_\varphi$, and $p_\psi^{\text{high}} > p_\psi$ we have that[5]

$$V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi, p_\psi^{\text{high}}) - V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^{\text{high}}, p_\psi^{\text{high}}) \geq V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi, p_\psi) - V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^{\text{high}}, p_\psi). \quad (15)$$

Suppose that $\varphi, \psi \in \Omega_{\to i}$.[6] There are three cases to consider:

Case 1: Suppose that $\varphi \notin \Phi$ for any $\Phi \in D_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi, p_\psi)$. Then, by individual rationality, $\varphi \notin \Phi$ for all $\Phi \in D_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^{\text{high}}, p_\psi)$. Hence,

$$V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi, p_\psi) - V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^{\text{high}}, p_\psi) = 0$$

and so equation (15) is satisfied, as the left side of (15) must be non-negative.

Case 2: Suppose $\varphi \in \Phi$ for all $\Phi \in D_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^{\text{high}}, p_\psi^{\text{high}})$. Then, by individual rationality, $\varphi \in \Phi$ for all $\Phi \in D_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi, p_\psi^{\text{high}})$. Hence,

$$V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi, p_\psi^{\text{high}}) - V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^{\text{high}}, p_\psi^{\text{high}}) = -(p_\varphi - p_\varphi^{\text{high}}) = p_\varphi^{\text{high}} - p_\varphi$$

and so equation (15) is satisfied, as the right side of (15) is (weakly) bounded from above by $p_\varphi^{\text{high}} - p_\varphi$ (with equality in the case that φ is demanded at both $(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi, p_\psi)$ and $(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^{\text{high}}, p_\psi)$).

Case 3: Suppose that $\varphi \in \Phi$ for some $\Phi \in D_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi, p_\psi)$ and $\varphi \notin \Phi$ for some $\Phi \in D_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^{\text{high}}, p_\psi^{\text{high}})$. In this case, as the preferences of i are fully substitutable,

[5]The definition of submodularity given in Definition 4 is equivalent to the pointwise definition given here; see, e.g., Schrijver (2002).

[6]The other three cases—

1. $\varphi \in \Omega_{\to i}$ and $\psi \in \Omega_{i \to}$,
2. $\varphi \in \Omega_{\to i}$ and $\psi \in \Omega_{i \to}$, and
3. $\varphi, \psi \in \Omega_{i \to}$—

are analogous.

there exists a unique price $p_\varphi^\uparrow$ such that there exists $\Phi, \bar{\Phi} \in D_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^\uparrow, p_\psi^{\text{high}})$ such that $\varphi \in \Phi$ and $\varphi \notin \bar{\Phi}$; note that $p_\varphi \le p_\varphi^\uparrow \le p_\varphi^{\text{high}}$. Similarly, let $p_\varphi^\downarrow$ be the unique price at which there exists $\Phi, \bar{\Phi} \in D_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^\downarrow, p_\psi)$ such that $\varphi \in \Phi$ and $\varphi \notin \bar{\Phi}$; note that $p_\varphi \le p_\varphi^\downarrow \le p_\varphi^{\text{high}}$. By the definition of the utility function, $\varphi \in \Phi$ for all $\Phi \in D_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, \tilde{p}_\varphi, p_\psi^{\text{high}})$ for all $\tilde{p}_\varphi < p_\varphi^\uparrow$, and $\varphi \notin \Phi$ for all $\Phi \in D_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, \tilde{p}_\varphi, p_\psi^{\text{high}})$ for all $\tilde{p}_\varphi > p_\varphi^\uparrow$; similarly, $\varphi \in \Phi$ for all $\Phi \in D_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, \tilde{p}_\varphi, p_\psi)$ for all $\tilde{p}_\varphi < p_\varphi^\downarrow$, and $\varphi \notin \Phi$ for all $\Phi \in D_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, \tilde{p}_\varphi, p_\psi)$ for all $\tilde{p}_\varphi > p_\varphi^\downarrow$.

Since the preferences of i are fully substitutable, $p_\varphi^\downarrow \le p_\varphi^\uparrow$. Hence,

$$
\begin{aligned}
&V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi, p_\psi^{\text{high}}) - V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^{\text{high}}, p_\psi^{\text{high}}) \\
&\quad = V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi, p_\psi^{\text{high}}) - V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^\uparrow, p_\psi^{\text{high}}) \\
&\qquad\quad + V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^\uparrow, p_\psi^{\text{high}}) - V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^{\text{high}}, p_\psi^{\text{high}}) \\
&\quad = -p_\varphi + p_\varphi^\uparrow - 0 \\
&\quad \ge -p_\varphi + p_\varphi^\downarrow - 0 \\
&\quad = V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi, p_\psi) - V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^\downarrow, p_\psi) \\
&\qquad\quad + V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^\downarrow, p_\psi) - V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^{\text{high}}, p_\psi) \\
&\quad = V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi, p_\psi) - V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^{\text{high}}, p_\psi),
\end{aligned}
$$

which is exactly (15).

Now, suppose that the preferences of i are not substitutable. We suppose moreover that the preferences of i fail the first condition of Defintion 2.[7] Hence, for some price vectors $p, p' \in \mathbb{R}^\Omega$ such that $|D_i(p)| = |D_i(p')| = 1$, $p_\omega = p'_\omega$ for all $\omega \in \Omega_{i\to}$, and $p_\omega \ge p'_\omega$ for all $\omega \in \Omega_{\to i}$, we have that for the unique $\Psi \in D_i(p)$ and $\Psi' \in D_i(p')$, either $\{\omega \in \Psi'_{\to i} : p_\omega = p'_\omega\} \not\subseteq \Psi_{\to i}$ or $\Psi_{i\to} \not\subseteq \Psi'_{i\to}$. We suppose that $\{\omega \in \Psi'_{\to i} : p_\omega = p'_\omega\} \not\subseteq \Psi_{\to i}$; the latter case is analogous. Let $\varphi \in \Psi_{\to i} \smallsetminus \{\omega \in \Psi'_{\to i} : p_\omega = p'_\omega\}$. Let p_φ^{high} be a price for trade φ high enough such that φ is

[7] The case where the preferences of i fail the second condition of Defintion 2 is analogous.

not demanded at either $(p_\varphi^{\text{high}}, p_{\Omega\smallsetminus\{\varphi\}})$ or $(p_\varphi^{\text{high}}, p'_{\Omega\smallsetminus\{\varphi\}})$. Hence,

$$V_i(p_\varphi, p'_{\Omega\smallsetminus\{\varphi\}}) - V_i(p_\varphi^{\text{high}}, p'_{\Omega\smallsetminus\{\varphi\}}) = 0$$

while

$$V_i(p_\varphi, p_{\Omega\smallsetminus\{\varphi\}}) - V_i(p_\varphi^{\text{high}}, p_{\Omega\smallsetminus\{\varphi\}}) > 0.$$

Thus we see that V_i is not submodular.

Proof of Theorem 7

The proof is an adaptation of the proof of Theorem 1 of Sun and Yang (2009) to our setting. As our model is more general than that of Sun and Yang (2009)—we do not impose either monotonicity or boundedness on the valuation functions, and we do not require that the seller values each bundle at 0 and thus sells everything that he could sell—we have to carefully ensure that the Sun and Yang (2009) approach remains valid.

We show first that IDFS and IIFS imply the single improvement property. Fix an arbitrary price vector $p \in \mathbb{R}^\Omega$ and a set of trades $\Psi \notin D_i(p)$ such that $u_i(\Psi) \neq -\infty$. Fix a set of trades $\Xi \in D_i(p)$. We focus exclusively on the trades in Ψ and Ξ by rendering all other trades that agent i is involved in irrelevant. To this end, we first define a very high price Π,

$$\Pi \equiv \max_{\substack{\Omega_1\subseteq\Omega_i, u_i(\Omega_1)>-\infty,\\ \Omega_2\subseteq\Omega_i, u_i(\Omega_2)>-\infty}} \{|U_i([\Omega_1; p]) - U_i([\Omega_2; p)|\} + \max_{\omega\in\Omega_i}\{|p_\omega|\} + 1,$$

and then, starting from p, we construct a preliminary price vector p' as follows:

$$p'_\omega = \begin{cases} p_\omega & \omega \in \Psi \cup \Xi \text{ or } \omega \notin \Omega_i \\ p_\omega + \Pi & \omega \in \Omega_{\to i} \smallsetminus (\Psi \cup \Xi) \\ p_\omega - \Pi & \omega \in \Omega_{i\to} \smallsetminus (\Psi \cup \Xi). \end{cases}$$

Observe that $\Psi \notin D_i(p')$ and $\Xi \in D_i(p')$. As $\Psi \neq \Xi$, we have to consider two cases (each with several subcases), which taken together will show that there exists a set of trades $\Phi' \neq \Psi$ that satisfies conditions 2 and 3 of Definition 5 and $U_i([\Phi'; p]) \geq U_i([\Psi; p])$.

Case 1: $\Xi \smallsetminus \Psi \neq \varnothing$**.** Select a trade $\xi_1 \in \Xi \smallsetminus \Psi$. Without loss of generality, assume that agent i is the buyer of ξ_1 (the case where i is the seller is completely analogous).

Starting from p', construct a modified price vector p'' as follows:

$$p''_\omega = \begin{cases} p'_\omega & \omega \in \Omega_i \smallsetminus ((\Xi_{\to i} \smallsetminus (\Psi_{\to i} \cup \{\xi_1\})) \cup \Psi_{i\to}) \text{ or } \omega \notin \Omega_i \\ p'_\omega + \Pi & \omega \in (\Xi_{\to i} \smallsetminus (\Psi_{\to i} \cup \{\xi_1\})) \cup \Psi_{i\to}. \end{cases}$$

First, since $\Xi \in D_i(p')$, $\xi_1 \in \Xi$, and $p'_{\xi_1} = p''_{\xi_1}$, full substitutability (Definition A.5) implies that there exists $\Xi'' \in D_i(p'')$ such that $\xi_1 \in \Xi''$. Second, observe that following the price change from p' to p'', $(\Xi''_{\to i} \smallsetminus \Psi_{\to i}) \subseteq \{\xi_1\}$ and $\Psi_{i\to} \subseteq \Xi''_{i\to}$. Thus, $\Xi''_{\to i} \smallsetminus \Psi_{\to i} = \{\xi_1\}$ and $\Psi_{i\to} \subseteq \Xi''_{i\to}$. We consider three subcases.

Subcase (a): $\Xi''_{i\to} \smallsetminus \Psi_{i\to} \neq \varnothing$**.** Let $\xi_2 \in \Xi''_{i\to} \smallsetminus \Psi_{i\to}$. Starting from p'', construct price vector p''' as follows:

$$p'''_\omega = \begin{cases} p''_\omega & \omega \in \Omega_i \smallsetminus ((\Xi_{i\to} \smallsetminus (\Psi_{i\to} \cup \{\xi_2\})) \cup \Psi_{\to i}) \text{ or } \omega \notin \Omega_i \\ p''_\omega - \Pi & \omega \in (\Xi_{i\to} \smallsetminus (\Psi_{i\to} \cup \{\xi_2\})) \cup \Psi_{\to i}. \end{cases}$$

First, since $\Xi'' \in D_i(p'')$, $\xi_2 \in \Xi''$, and $p''_{\xi_2} = p'''_{\xi_2}$, full substitutability (Definition A.6) implies that there exists $\Xi''' \in D_i(p''')$ such that $\xi_2 \in \Xi'''$. Second, observe that following the price change from p'' to p''', $\Psi \subseteq \Xi'''$ and $\Xi''' \smallsetminus \Psi \subseteq \{\xi_1, \xi_2\}$. Thus, $\Psi \smallsetminus \Xi''' = \varnothing$ and $\Xi''' \smallsetminus \Psi = \{\xi_1, \xi_2\}$ or $\{\xi_2\}$.

Since $\Xi''' \in D_i(p''')$, we have $U_i([\Psi, p''']) \leq U_i([\Xi''', p'''])$. Furthermore, observe that from the perspective of agent i the only differences from Ψ to Ξ''' are making one new sale ξ_2, i.e., $e_{i,\xi_2}(\Psi) > e_{i,\xi_2}(\Xi''')$ with $\xi_2 \in \Omega_{i\rightarrow} \smallsetminus \Psi$, and (possibly) making one new purchase ξ_1, i.e. $e_{i,\xi_1}(\Psi) < e_{i,\xi_1}(\Xi''')$ with $\xi_1 \in \Omega_{\rightarrow i} \smallsetminus \Psi$.

Subcase (b): $\Xi''_{i\rightarrow} \smallsetminus \Psi_{i\rightarrow} = \varnothing$ and $\Psi_{\rightarrow i} \smallsetminus \Xi''_{\rightarrow i} \neq \varnothing$. Let $\psi \in \Psi_{\rightarrow i} \smallsetminus \Xi''_{\rightarrow i}$. Starting from p'', construct price vector p''' as follows:

$$p'''_\omega = \begin{cases} p''_\omega & \omega \in \Omega_i \smallsetminus ((\Xi_{i\rightarrow} \smallsetminus \Psi_{i\rightarrow}) \cup (\Psi_{\rightarrow i} \smallsetminus \{\psi\})) \text{ or } \omega \notin \Omega_i \\ p''_\omega - \Pi & \omega \in (\Xi_{i\rightarrow} \smallsetminus \Psi_{i\rightarrow}) \cup (\Psi_{\rightarrow i} \smallsetminus \{\psi\}). \end{cases}$$

First, since $\Xi'' \in D_i(p'')$, $\psi \notin \Xi''$, and $p''_\psi = p'''_\psi$, by full substitutability (Definition A.6) implies that there exists $\Xi''' \in D_i(p''')$ such that $\psi \notin \Xi'''$. Second, observe that following the price change from p'' to p''', $\Psi \smallsetminus \Xi''' \subseteq \{\psi\}$ and $\Xi''' \smallsetminus \Psi \subseteq \{\xi_1\}$. Thus, $\Psi \smallsetminus \Xi''' = \{\psi\}$ and $\Xi''' \smallsetminus \Psi = \{\xi_1\}$ or $\varnothing$.

Since $\Xi''' \in D_i(p''')$, we have $U_i([\Psi, p''']) \leq U_i([\Xi''', p'''])$. Furthermore, observe that from agent i's perspective the only differences from Ψ to Ξ''' are canceling one purchase ψ, i.e., $e_{i,\psi}(\Psi) > e_{i,\psi}(\Xi''')$ with $\psi \in \Psi_{\rightarrow i}$, and (possibly) making one new purchase ξ_1, i.e., $e_{i,\xi_1}(\Psi) < e_{i,\xi_1}(\Xi''')$ with $\xi_1 \in \Omega_{\rightarrow i} \smallsetminus \Psi$.

Subcase (c): $\Xi'' = \Psi \cup \{\xi_1\}$. Let $p''' = p''$ and $\Xi''' = \Xi''$. Since $\Xi''' \in D_i(p''')$, we have $U_i([\Psi, p''']) \leq U_i([\Xi''', p'''])$. Furthermore, observe that from agent i's perspective the only difference from Ψ to Ξ''' is making a new purchase ξ_1, i.e., $e_{i,\xi_1}(\Psi) < e_{i,\xi_1}(\Xi''')$ with $\xi_1 \in \Omega_{\rightarrow i} \smallsetminus \Psi$.

Case 2: $\Xi \smallsetminus \Psi = \varnothing$ and $\Psi \smallsetminus \Xi \neq \varnothing$. Select a trade $\psi_1 \in \Psi \smallsetminus \Xi$. Without loss of generality, assume that agent i is a buyer in ψ_1 (the case where i is a seller is completely analogous).

Starting from p', construct price vector p'' as follows:

$$p''_\omega = \begin{cases} p'_\omega & \omega \in \Omega_i \smallsetminus (\Psi_{\rightarrow i} \smallsetminus \{\psi_1\}) \text{ or } \omega \notin \Omega_i \\ p'_\omega - \Pi & \omega \in \Psi_{\rightarrow i} \smallsetminus \{\psi_1\}. \end{cases}$$

First, since $\Xi \in D_i(p')$, $\psi_1 \notin \Xi$, and $p'_{\psi_1} = p''_{\psi_1}$, full substitutability (Definition A.6) implies that there exists $\Xi'' \in D_i(p'')$ such that $\psi_1 \notin \Xi''$. Second, observe that following the price change from p' to p'', $\Xi'' \subseteq \Psi$ and $\Psi_{\to i} \smallsetminus \Xi''_{\to i} \subseteq \{\psi_1\}$. Thus, $\Psi_{\to i} \smallsetminus \Xi''_{\to i} = \{\psi_1\}$ and $\Xi'' \subseteq \Psi$. We consider two subcases.

Subcase (a): $\Psi_{i\to} \smallsetminus \Xi''_{i\to} \neq \varnothing$. Let $\psi_2 \in \Psi_{i\to} \smallsetminus \Xi''_{i\to}$. Starting from p'', construct price vector p''' as follows:

$$p'''_\omega = \begin{cases} p''_\omega & \omega \in \Omega_i \smallsetminus (\Psi_{i\to} \smallsetminus \{\psi_2\}) \text{ or } \omega \notin \Omega_i \\ p''_\omega + \Pi & \omega \in \Psi_{i\to} \smallsetminus \{\psi_2\}. \end{cases}$$

First, since $\Xi'' \in D_i(p'')$, $\psi_2 \notin \Xi''$, and $p''_{\psi_2} = p'''_{\psi_2}$, full substitutability (definition A.5) implies that there exists $\Xi''' \in D_i(p''')$ such that $\psi_2 \notin \Xi'''$. Second, observe that following the price change from p'' to p''', $\Xi''' \subseteq \Psi$ and $\Psi \smallsetminus \Xi''' \subseteq \{\psi_1, \psi_2\}$. Thus, $\Xi''' \smallsetminus \Psi = \varnothing$ and $\Psi \smallsetminus \Xi''' = \{\psi_1, \psi_2\}$ or $\{\psi_2\}$.

Since $\Xi''' \in D_i(p''')$, we have $U_i([\Psi, p''']) \leq U_i([\Xi''', p'''])$. Furthermore, observe that from agent i's perspective the only differences from Ψ to Ξ''' are canceling one sale ψ_2, i.e., $e_{i,\psi_2}(\Psi) < e_{i,\psi_2}(\Xi''')$ with $\psi_1 \in \Omega_{i\to} \smallsetminus \Psi$, and (possibly) canceling one purchase ψ_1, i.e., $e_{i,\psi_1}(\Psi) > e_{i,\psi_1}(\Xi''')$ with $\psi_1 \in \Psi_{\to i}$.

Subcase (b): $\Xi'' = \Psi \smallsetminus \{\psi_1\}$. In this subcase, let $p''' = p''$ and $\Xi''' = \Xi''$. Since $\Xi''' \in D_i(p''')$, we have $U_i([\Psi, p''']) \leq U_i([\Xi''', p'''])$. Furthermore, observe that from the perspective of agent i, the only difference from Ψ to Ξ''' is canceling purchase ψ_1, i.e., $e_{i,\psi_1}(\Psi) < e_{i,\psi_1}(\Xi''')$ with $\psi_1 \in \Omega_{\to i} \smallsetminus \Psi$.

Taking together all the final statements from each subcase of Cases 1 and 2, if we take $\Phi' \equiv \Xi'''$, we obtain that we always have a price vector p''' and the sets Ψ and Φ' that satisfy conditions (2) and (3) of Definition 5. Moreover, since we always have $\Phi \in D_i(p''')$, $U_i([\Phi', p''']) \geq U_i([\Psi, p'''])$.

Next, we show that $U_i([\Phi', p''']) - U_i([\Psi, p''']) \geq 0$ implies $U_i([\Phi', p]) \geq U_i([\Psi, p])$. First, observe that when taking the difference the prices of all trades $\omega \in \Phi' \cap \Psi$ cancel each other out. Thus, replacing the prices p'''_ω with p_ω for all trades $\omega \in \Phi' \cap \Psi$ leaves the difference unchanged. Second, observe that in all previous subcases, the construction of p''' implies that for all $\omega \in ((\Psi \setminus \Phi') \cup (\Phi' \setminus \Psi))$, $p_\omega = p'''_\omega$. Combining the two observations above, $U_i([\Phi', p''']) - U_i([\Psi, p''']) = U_i([\Phi', p]) - U_i([\Psi, p])$, and therefore $U_i([\Phi', p]) \geq U_i([\Psi, p])$.

We now show that there exists a set of trades Φ that satisfies all conditions of Definition 5. Since $\Psi \notin D_i(p)$, $V_i(p) > U_i([\Psi; p])$. Since i's utility is continuous in prices, there exists $\varepsilon > 0$ such that $V_i(q) > U_i([\Psi; q])$ where q is defined as follows:

$$q_\omega = \begin{cases} p_\omega + \varepsilon & \omega \in (\Omega_{\to i} \setminus \Psi_{\to i}) \cup \Psi_{i \to} \\ p_\omega - \varepsilon & \omega \in (\Omega_{i \to} \setminus \Psi_{i \to}) \cup \Psi_{\to i}. \end{cases}$$

Our arguments above imply that there exists a set of trades $\Phi \neq \Psi$ such that $U_i([\Phi; q]) \geq U_i([\Psi; q])$. Using the construction of q, we obtain $U_i([\Phi; p]) - U_i([\Psi; p]) = U_i([\Phi; q]) - U_i([\Psi; q]) + \varepsilon |(\Psi \setminus \Phi) \cup (\Phi \setminus \Psi)| > U_i([\Phi; q]) - U_i([\Psi; q]) \geq 0$. Thus, $U_i([\Phi; p]) > U_i([\Psi; p])$. This completes the proof that IDFS and IIFS imply the single improvement property.

We now show that the single improvement property implies full substitutability DCFS. More specifically, we will establish that single improvement implies the first condition of Definition A.4. The proof that the second condition of Definition A.4 is also satisfied uses a completely analogous argument.

Let $p \in \mathbb{R}^\Omega$ and $\Psi \in D_i(p)$ be arbitrary. It is sufficient to establish that for any $p' \in \mathbb{R}^\Omega$ such that $p'_\psi > p_\psi$ for some $\psi \in \Omega_{\to i}$ and $p'_\omega = p_\omega$ for all $\omega \in \Omega \setminus \{\psi\}$, there exists a set of trades $\Psi' \in D_i(p')$ that satisfies the first condition of Definition A.4.

Fix one $p' \in \mathbb{R}^\Omega$ that satisfies the conditions mentioned in the previous paragraph and let $\psi \in \Omega_{\to i}$ be the one trade for which $p'_\psi > p_\psi$. Note that if either $\psi \notin \Psi$ or $\Psi \in D_i(p')$, there is nothing to show. From now on, assume that $\psi \in \Psi$ and $\Psi \notin D_i(p')$.

For any real number $\varepsilon > 0$ define a price vector $p^\varepsilon \in \mathbb{R}^\Omega$ by setting $p^\varepsilon_\psi = p_\psi + \varepsilon$ and $p^\varepsilon_\omega = p_\omega$ for all $\omega \in \Omega \setminus \{\psi\}$. Let $\Delta \equiv \max\{\varepsilon : \Psi \in D_i(p^\varepsilon)\}$. Note that Δ is well defined since i's utility function is continuous in prices. Furthermore, given that $\Psi \notin D_i(p')$, we must have $\Delta < p'_\psi - p_\psi$.

Next, for any integer n, define a price vector $p^n \in \mathbb{R}^\Omega$ by setting $p^n_\psi = p_\psi + \Delta + \frac{1}{n}$ and $p^n_\omega = p_\omega$ for all $\omega \in \Omega \setminus \{\psi\}$. By the definition of Δ we must have $\Psi \notin D_i(p^n)$ for all $n > 0$. By the single improvement property, this implies that for all $n > 0$, there exists a set of trades Φ^n such that the following conditions are satisfied:

1. $U_i([\Psi, p^n]) < U_i([\Phi^n, p^n])$,

2. there exists at most one trade ω such that $e_{i,\omega}(\Psi) < e_{i,\omega}(\Phi^n)$, and

3. there exists at most one trade ω such that $e_{i,\omega}(\Psi) > e_{i,\omega}(\Phi^n)$.

Note that we must have $\psi \notin \Phi^n$ for all $n \geq 1$. This follows since for any $n \geq 1$ and any set of trades Φ such that $\psi \in \Phi$, $U_i([\Phi; p^n]) = U_i([\Phi; p]) - \Delta - \frac{1}{n} \leq U_i([\Psi; p]) - \Delta - \frac{1}{n} = U_i([\Psi; p^n])$ given that $\Psi \in D_i(p)$.

Conditions 2 and 3 imply that for all $n > 0$, we must have $\{\omega \in \Psi_{\to i} : p'_\omega = p_\omega\} = \{\omega \in \Psi_{\to i} : p^n_\omega = p_\omega\} \subseteq \Phi^n_{\to i}$ and $\Phi^n_{i\to} \subseteq \Psi_{i\to}$.

Since the set of trades is finite, it is without loss of generality to assume that there is a set of trades $\Phi^* \in \Omega_i$ and an integer $\bar{n}$ such that $\Phi^n = \Phi^*$ for all $n \geq \bar{n}$. Since i's utility function is continuous with respect to prices and $p^n \to p^\Delta$, we must have $U_i([\Phi^*; p^\Delta]) \geq U_i([\Psi; p^\Delta])$. Since $\Psi \in D_i(p^\Delta)$, this implies $\Phi^* \in D_i(p^\Delta)$. Since $\Delta < p'_\psi - p_\psi$ and V_i is decreasing in the prices of trades for which i is a buyer, we must have $V_i(p^\Delta) \geq V_i(p')$. Since $\psi \notin \Phi^*$, we have that $U_i([\Phi^*; p']) = U_i([\Phi^*; p^\Delta]) = V_i(p^\Delta)$. Hence, $\Phi^* \in D_i(p')$ and setting $\Psi' \equiv \Phi^*$ yields a set that satisfies the first condition of Definition A.4.

Proof of Theorem 8

The proof is an adaptation of the proof of Theorem 1 of Gul and Stacchetti (1999). Since we impose neither monotonicity nor boundedness conditions on valuation functions, there are a number of details needed in order to check that Gul and Stacchetti (1999) proof strategy works in our setting.

Throughout the proof, for any price vector $p \in \mathbb{R}^{\Omega}$, we denote by $\tilde{D}_i(p)$ the sets of objects that correspond to the optimal sets of trades in $D_i(p)$.

We show first that the single improvement property in object-language implies the no complementarities condition. Let p be an arbitrary price vector and $\mathbf{\Phi}, \mathbf{\Psi} \in \tilde{D}_i(p)$ be arbitrary. Let $\bar{\mathbf{\Psi}} \subseteq \mathbf{\Psi} \smallsetminus \mathbf{\Phi}$ be arbitrary. Let $\mathbf{\Xi} \in \tilde{D}_i(p)$ be a set of objects such that $\mathbf{\Xi} \subseteq \mathbf{\Phi} \cup \mathbf{\Psi}$ and $\mathbf{\Psi} \smallsetminus \bar{\mathbf{\Psi}} \subseteq \mathbf{\Xi}$, and such that there is no $\mathbf{\Xi}' \in \tilde{D}_i(p)$ for which $\mathbf{\Xi}' \subseteq \mathbf{\Phi} \cup \mathbf{\Psi}$, $\mathbf{\Psi} \setminus \bar{\mathbf{\Psi}} \subseteq \mathbf{\Xi}'$, and $|\mathbf{\Xi}' \cap \bar{\mathbf{\Psi}}| < |\mathbf{\Xi} \cap \bar{\mathbf{\Psi}}|$. If $\mathbf{\Xi} \cap \bar{\mathbf{\Psi}} = \varnothing$, we are done. If not, let Π be a very large number[8] and define $p(\varepsilon)$ by setting $p_{\mathsf{t}(\boldsymbol{\omega})}(\varepsilon) = \Pi$ if $\boldsymbol{\omega} \in \mathbf{\Omega}_{\to i} \smallsetminus (\mathbf{\Phi} \cup \mathbf{\Psi})$, $p_{\mathsf{t}(\boldsymbol{\omega})}(\varepsilon) = -\Pi$ if $\boldsymbol{\omega} \in \mathbf{\Omega}_{i \to} \smallsetminus (\mathbf{\Phi} \cup \mathbf{\Psi})$, $p_{\mathsf{t}(\boldsymbol{\omega})}(\varepsilon) = p_{\mathsf{t}(\boldsymbol{\omega})}$ if $\boldsymbol{\omega} \in (\mathbf{\Phi} \cup \mathbf{\Psi}) \smallsetminus \bar{\mathbf{\Psi}}$, and $p_{\mathsf{t}(\boldsymbol{\omega})}(\varepsilon) = p_{\mathsf{t}(\boldsymbol{\omega})} + \varepsilon$ if $\boldsymbol{\omega} \in \bar{\mathbf{\Psi}}$. Note that for all $\varepsilon > 0$ we must have $\mathbf{\Phi} \in \tilde{D}_i(p(\varepsilon))$ (since $\bar{\mathbf{\Psi}} \subseteq \mathbf{\Psi} \setminus \mathbf{\Phi}$) and $U_i([\mathbf{\Phi}; p(\varepsilon)]) > U_i([\mathbf{\Xi}; p(\varepsilon)])$. Since $\mathbf{\Xi} \in \tilde{D}_i(p)$, we must have $u_i(\mathbf{\Xi}) \neq -\infty$. Hence, we can apply the single improvement property (in object-language) to infer that there must exist a set of objects $\mathbf{\Xi}'$ such that $|\mathbf{\Xi}' \setminus \mathbf{\Xi}| \leq 1$, $|\mathbf{\Xi} \setminus \mathbf{\Xi}'| \leq 1$, and $U_i([\mathbf{\Xi}'; p(\varepsilon)]) > U_i([\mathbf{\Xi}; p(\varepsilon)])$. Given the definition of $p(\varepsilon)$ and Π, we must have $\mathbf{\Xi}' \subseteq \mathbf{\Phi} \cup \mathbf{\Psi}$. Since $U_i([\mathbf{\Xi}'; p(\varepsilon)]) > U_i([\mathbf{\Xi}; p(\varepsilon)])$ holds for arbitrarily small values of ε, we must have $\mathbf{\Xi}' \in \tilde{D}_i(p)$. But $U_i([\mathbf{\Xi}'; p(\varepsilon)]) > U_i([\mathbf{\Xi}; p(\varepsilon)])$ is equivalent to $U_i([\mathbf{\Xi}'; p]) - |\mathbf{\Xi}' \cap \bar{\mathbf{\Psi}}|\varepsilon > U_i([\mathbf{\Xi}; p]) - |\mathbf{\Xi} \cap \bar{\mathbf{\Psi}}|\varepsilon$. Given that $\mathbf{\Xi}, \mathbf{\Xi}' \in \tilde{D}_i(p)$, the last inequality is equivalent to $|\mathbf{\Xi}' \cap \bar{\mathbf{\Psi}}| < |\mathbf{\Xi} \cap \bar{\mathbf{\Psi}}|$ and we thus obtain a contradiction to the definition of $\mathbf{\Xi}$. Hence, it has to be the case that $\mathbf{\Xi} \cap \bar{\mathbf{\Psi}} = \varnothing$ and this completes the proof that single improvement implies no complementarities.

[8]For instance, let

$$\Delta = \max_{\Omega_1 \subset \Omega_i, \Omega_2 \subset \Omega_i, u_i(\Omega_1) > -\infty, u_i(\Omega_2) > -\infty} |U_i([\Omega_1; p]) - U_i([\Omega_2; p])|,$$

and $\Pi = 1 + \Delta + \max_{\omega \in \Omega_i} |p_\omega|$.

Next, we show that the generalized no complementarities condition implies object-language full substitutability. Let p, p' be two price vectors such that $p \le p'$. Let $\mathbf{\Psi} \in \tilde{D}_i(p)$ be arbitrary.[9] Let $\tilde{\mathbf{\Omega}}_i = \{\boldsymbol{\omega} \in \mathbf{\Omega}_i : p_{\mathrm{t}(\boldsymbol{\omega})} < p'_{\mathrm{t}(\boldsymbol{\omega})}\}$. The proof will proceed by induction on $|\tilde{\mathbf{\Omega}}_i|$. Consider first the case of $|\tilde{\mathbf{\Omega}}_i| = 1$ and let $\tilde{\mathbf{\Omega}}_i = \{\boldsymbol{\omega}\}$. Clearly, if $\boldsymbol{\omega} \notin \mathbf{\Psi}$ or $\mathbf{\Psi} \in \tilde{D}_i(p')$, there is nothing to show. So suppose that $\boldsymbol{\omega} \notin \mathbf{\Psi}$ and that $\mathbf{\Psi} \notin \tilde{D}_i(p')$. For any $\varepsilon \ge 0$, define a price vector $p(\varepsilon)$ by setting $p_{\mathrm{t}(\boldsymbol{\varphi})}(\varepsilon) = p_{\mathrm{t}(\boldsymbol{\varphi})}$ for all $\boldsymbol{\varphi} \neq \boldsymbol{\omega}$, and $p_{\mathrm{t}(\boldsymbol{\omega})}(\varepsilon) = p_{\mathrm{t}(\boldsymbol{\omega})} + \varepsilon$. Let $\bar{\varepsilon} = \max\{\varepsilon : \mathbf{\Psi} \in \tilde{D}_i(p(\varepsilon))\}$ and note that $\bar{\varepsilon} < p'_{\mathrm{t}(\boldsymbol{\omega})} - p_{\mathrm{t}(\boldsymbol{\omega})}$ given that $\mathbf{\Psi} \notin \tilde{D}_i(p')$. Consider some $\varepsilon > \bar{\varepsilon}$ and fix a set of objects $\mathbf{\Phi} \in \tilde{D}_i(p(\varepsilon))$. It is easy to see that $\boldsymbol{\omega} \notin \mathbf{\Phi}$ and that $\mathbf{\Phi} \in \tilde{D}_i(p(\bar{\varepsilon}))$. By the generalized no complementarities condition, there must exist a set of objects $\mathbf{\Xi} \subseteq \mathbf{\Phi}$ such that $\mathbf{\Psi}' := \mathbf{\Psi} \setminus \{\boldsymbol{\omega}\} \cup \Xi \in \tilde{D}_i(p(\bar{\varepsilon}))$. Clearly, we must also have $\mathbf{\Psi}' \in \tilde{D}_i(p')$ and this completes the proof in case of $|\tilde{\mathbf{\Omega}}_i| = 1$. Now suppose that the statement has already been established for all pairs of price vectors p, p' such that $|\tilde{\mathbf{\Omega}}_i| \le K$ for some $K \ge 1$. Consider two price vectors p, p' such that $|\tilde{\mathbf{\Omega}}_i| = K + 1$. Fix a set of objects $\mathbf{\Psi} \in \tilde{D}_i(p)$. Let $\boldsymbol{\omega} \in \tilde{\mathbf{\Omega}}_i$ be arbitrary and consider a price vector p'' such that $p''_{\mathrm{t}(\boldsymbol{\omega})} = p_{\mathrm{t}(\boldsymbol{\omega})}$ and $p''_{\mathrm{t}(\boldsymbol{\varphi})} = p'_{\mathrm{t}(\boldsymbol{\varphi})}$ for all $\boldsymbol{\varphi} \neq \boldsymbol{\omega}$. By the inductive assumption, there is a set $\mathbf{\Psi}'' \in \tilde{D}_i(p'')$ such that $\{\boldsymbol{\varphi} \in \mathbf{\Psi} : p''_{\mathrm{t}(\boldsymbol{\varphi})} = p_{\mathrm{t}(\boldsymbol{\varphi})}\} \subseteq \mathbf{\Psi}''$. Note that $\{\boldsymbol{\varphi} \in \mathbf{\Psi} : p'_{\mathrm{t}(\boldsymbol{\varphi})} = p_{\mathrm{t}(\boldsymbol{\varphi})}\} = \{\boldsymbol{\varphi} \in \mathbf{\Psi} : p''_{\mathrm{t}(\boldsymbol{\varphi})} = p_{\mathrm{t}(\boldsymbol{\varphi})}\} \setminus \{\boldsymbol{\omega}\}$. Applying the inductive assumption one more time, there has to be a set $\mathbf{\Psi}' \in \tilde{D}_i(p')$ such that $\mathbf{\Psi}'' \setminus \{\boldsymbol{\omega}\} \subseteq \mathbf{\Psi}'$. Combining this with the previous arguments, we obtain $\{\boldsymbol{\varphi} \in \mathbf{\Psi} : p'_{\mathrm{t}(\boldsymbol{\varphi})} = p_{\mathrm{t}(\boldsymbol{\varphi})}\} \subseteq \mathbf{\Psi}'$. This completes the proof.

Proof of Theorem 9

As Ω is finite and non-empty, for each agent i the domain of u_i is bounded and non-empty. Hence, by Part (b) of Theorem 7 of Murota and Tamura (2003), we see that u_i is $M^\natural$-concave over objects if and only if the preferences of i have the single-improvement property.[10] The

[9] There is no need to rule out the possibility of several optimal bundles of objects in this proof.

[10] Strictly speaking, Theorem 7(b) shows the equivalence of $M^\natural$-convexity and the ($\mathrm{M}^\natural$-SI) property of a function f. It is, however, immediate that this result implies the equivalence of $M^\natural$-concavity and the

result then follows from Theorem 7.

Proof of Theorem 10

We assume throughout that $\Omega = \Omega_i$ (and so $X = X_i$); this is without loss of generality as all of the analysis here considers only the sets of contracts demanded by i and, for any sets of contracts Y and Z such that $Y_i = Z_i$, we have that $Y^* \in C_i(Y)$ if and only if $Y^* \in C_i(Z)$.

Step 1: We show first that full substitutability implies monotone–substitutability for opportunity sets such that the choice correspondence is single-valued. That is, we will show for all finite sets of contracts Y and Z such that $|C_i(Y)| = |C_i(Z)| = 1$, $Y_{i\to} = Z_{i\to}$, and $Y_{\to i} \subseteq Z_{\to i}$, for the unique $Y^* \in C_i(Y)$ and the unique $Z^* \in C_i(Z)$, we have $|Z^*_{\to i}| - |Y^*_{\to i}| \geq |Z^*_{i\to}| - |Y^*_{i\to}|$.

Fix a fully substitutable valuation function u_i for agent i. Consider two finite sets of contracts Y and Z such that $|C_i(Y)| = |C_i(Z)| = 1$, $Y_{i\to} = Z_{i\to}$, and $Y_{\to i} \subseteq Z_{\to i}$. Assume that for any $\omega \in \Omega_{i\to}$, if $(\omega, p_\omega) \in Y_{i\to}$ and $(\omega, p'_\omega) \in Y_{i\to}$, then $p_\omega = p'_\omega$; this is without loss of generality, because for a given trade $\omega \in \Omega_{i\to}$, agent i, as a seller, will only choose a contract with the highest price available for that trade, and thus we can disregard all other contracts involving that trade.

Let $Y^* \in C_i(Y)$ and $Z^* \in C_i(Z)$. Define a modified valuation $\tilde{u}_i$ on $\tau(Z_i)$ for agent i by setting, for each $\Psi \subseteq \tau(Z_i)$,

$$\tilde{u}_i(\Psi) = u_i(\Psi_{\to i} \cup (\tau(Z) \smallsetminus \Psi)_{i\to}).$$

For all feasible $W \subseteq Z$, let

$$\tilde{U}_i(W) = \tilde{u}_i(\tau(W)) + \sum_{(\omega,p_\omega)\in(Z\smallsetminus W)_{i\to}} p_\omega - \sum_{(\omega,p_\omega)\in W_{\to i}} p_\omega,$$

single-improvement property for a function $g = -f$.

and let $\tilde{C}_i$ denote the choice correspondence over Z associated with $\tilde{U}_i$. By construction,

$$\tilde{u}_i(\Psi) = u_i(\tilde{\mathfrak{o}}_i(\Psi)), \tag{16}$$

where here the object operator $\tilde{\mathfrak{o}}$ is defined with respect to the underlying set of trades $\tau(Z)$:

$$\tilde{\mathfrak{o}}_i(\Psi) = \{\mathfrak{o}(\omega) : \omega \in \Psi_{\to i}\} \cup \{\mathfrak{o}(\omega) : \omega \in \tau(Z) \smallsetminus \Psi_{i\to}\}.$$

As the preferences of i are fully substitutable, the restriction of those preferences to $\tau(Z)$ is fully substitutable, as well.[11] Thus, the restriction of i's preferences to $\tau(Z)$ is object-language fully substitutable and so $\tilde{u}_i$ satisfies the gross substitutability condition of Kelso and Crawford (1982) over objects.

Now, we must have $\tilde{C}_i(Y) = \{Y^*_{\to i} \cup (Z \smallsetminus Y^*)_{i\to}\}$ and $\tilde{C}_i(Z) = \{Z^*_{\to i} \cup (Z \smallsetminus Z^*)_{i\to}\}$. As we assume quasilinearity, the Law of Aggregate Demand for two-sided markets applies to $\tilde{C}_i$ (by Theorem 7 of Hatfield and Milgrom (2005)). As $Y \subseteq Z$, this implies that $|Z^*_{\to i} \cup (Z \smallsetminus Z^*)_{i\to}| \geq |Y^*_{\to i} \cup (Z \smallsetminus Y^*)_{i\to}|$; this inequality is equivalent to $|Z^*_{\to i}| - |Z^*_{i\to}| \geq |Y^*_{\to i}| - |Y^*_{i\to}|$, which is precisely the Law of Aggregate Demand. We also immediately have that $Y_{i\to} \smallsetminus Y^*_{\to i} \subseteq Z_{i\to} \smallsetminus Z^*_{\to i}$ and $Y^*_{i\to} \subseteq Z^*_{i\to}$, as the preferences of i are fully substitutable. Thus the preferences of i satisfy the requirements of Part 1 of Definition 11 when the choice correspondence is single-valued.

The proof that the preferences of i satisfy the requirements of Part 2 of Definition 11 when the choice correspondence is single-valued is analogous. Thus, combining the preceding results, we obtain that full substitutability implies monotone–substitutability for opportunity sets such that the choice correspondence is single-valued.

Step 2: We now use Step 1 to show that full substitutability implies monotone–substitutability.

[11] To see the full substitutability of $\tilde{U}_i$, note that the full substitutability of the restriction of U_i to any subset of X follows immediately from the fact that the preferences of i satisfy CFS.

For this step, let

$$\hat{u}(\Psi; Y) \equiv u_i(\Psi) - \sum_{\psi \in \Psi_{\to i}} \inf\{p_\psi : (\psi, p_\psi) \in Y\} + \sum_{\psi \in \Psi_{i\to}} \sup\{p_\psi : (\psi, p_\psi) \in Y\},$$

where we take $\inf \varnothing = \infty$ and $\sup \varnothing = -\infty$; that is, $\hat{u}(\Psi; Y)$ is the utility that i obtains from the set of trades Ψ and both paying, for each trade in $\Psi_{\to i}$, the lowest price corresponding to a contract in Y and receiving, for each trade in $\Psi_{i\to}$, the highest price corresponding to a contract in Y.

We also extend the operator τ to sets of sets of contracts, so that $\tau(\mathcal{Y}) = \cup_{Y \in \mathcal{Y}}\{\tau(Y)\}$ for any $\mathcal{Y} \subseteq \wp(X)$.

Finally, it is helpful to define an operator which, given a set of available contracts W, makes each trade in $\tau(W')$ slightly more appealing to i relative to W' and each trade not in $\tau(W')$ slightly less appealing to i relative to W'. Let

$$\begin{aligned} r(W; W', \varepsilon) \equiv\ & \{(\omega, p_\omega - \varepsilon) \in X : (\omega, p_\omega) \in W'_{\to i}\} \\ & \cup \{(\omega, p_\omega + \varepsilon) \in X : (\omega, p_\omega) \in [W \smallsetminus W']_{\to i}\} \\ & \cup \{(\omega, p_\omega + \varepsilon) \in X : (\omega, p_\omega) \in W'_{i\to}\} \\ & \cup \{(\omega, p_\omega - \varepsilon) \in X : (\omega, p_\omega) \in [W \smallsetminus W']_{i\to}\}. \end{aligned}$$

The r function here allows us to perturb sets of contracts so as to obtain unique choices, similar to the methods used to prove Lemma 1.

Observation 1. *For all sets of contracts $W, Y, Z \subseteq X$ such that $Y \subseteq Z$, we have that $r(Y; W, \varepsilon) \subseteq r(Z; W, \varepsilon)$ for all $\varepsilon > 0$.*

Now, we consider two finite sets of contracts Y and Z such that $Y_{i\to} = Z_{i\to}$ and $Y_{\to i} \subseteq Z_{\to i}$. Fix an arbitrary $Y^* \in C_i(Y)$; we need to show that there exists a set $Z^* \in C_i(Z)$ that satisfies the requirements of Part 1 of Definition 11. Let $\hat{Z}^* \in C_i(r(Z; Y^*, \varepsilon))$.

We first show five intermediate results on the effects of our price perturbations, where we $\varepsilon > 0$ to be sufficiently small and $\delta > 0$ to be sufficiently small given ε.

Fact 1: $C_i(r(Y; Y^*, \varepsilon)) = \{r(Y^*; Y^*, \varepsilon)\}$. We have that, for any feasible $W \subseteq Y$ such that $W \neq Y^*$,[12,13]

$$\begin{aligned} U_i(r(Y^*; Y^*, \varepsilon)) - U_i(r(W; Y^*, \varepsilon)) &= U_i(Y^*) - U_i(W) + |Y^* \ominus W|\varepsilon \\ &\geq |Y^* \ominus W|\varepsilon \\ &> 0, \end{aligned}$$

where the equality follows from the definition of r, the first inequality follows from the fact that Y^* is optimal at Y (i.e., $Y^* \in C_i(Y)$) and the second inequality follows from the fact that $W \neq Y^*$. Thus, we see that $C_i(r(Y; Y^*, \varepsilon)) = \{r(Y^*; Y^*, \varepsilon)\}$, as desired.

Fact 2: $\tau(C_i(r(r(Y; Y^*, \varepsilon); \hat{Z}^*, \delta))) \subseteq \tau(C_i(r(Y; Y^*, \varepsilon)))$. Consider an arbitrary $\Phi \in \tau(C_i(r(Y; Y^*, \varepsilon)))$ and an arbitrary $\Xi \notin \tau(C_i(r(Y; Y^*, \varepsilon)))$. For ε small enough, we have that,

$$\begin{aligned} &\hat{u}(\Phi; r(r(Y; Y^*, \varepsilon); \hat{Z}^*, \delta)) - \hat{u}(\Xi; r(r(Y; Y^*, \varepsilon); \hat{Z}^*, \delta)) \\ &\quad \geq \hat{u}(\Phi; r(Y; Y^*, \varepsilon)) - \hat{u}(\Xi; r(Y; Y^*, \varepsilon)) - |\Phi \ominus \Xi|\delta \\ &\quad > 0, \end{aligned}$$

where the first inequality follows from the definition of r and the second inequality follows as Φ is associated with an optimal set of contracts at $r(Y; Y^*, \varepsilon)$, Ξ is not associated with an optimal set of contracts at $r(Y; Y^*, \varepsilon)$, and δ is sufficiently small. Thus, $\Xi \notin \tau(C_i(r(r(Y; Y^*, \varepsilon); \hat{Z}^*, \delta)))$ and so $\tau(C_i(r(r(Y; Y^*, \varepsilon); \hat{Z}^*, \delta))) \subseteq$

[12] Note that there is a natural one-to-one correspondence between (feasible) subsets of Y and (feasible) subsets of $r(Y; Y^*, \varepsilon)$.

[13] Here, we use $\ominus$ to denote the symmetric difference between two sets, i.e., $W \ominus W' = (W \smallsetminus W') \cup (W \smallsetminus W')$.

$\tau(C_i(r(Y;Y^*,\varepsilon)))$.

Fact 3: $\tau(C_i(r(Z;Y^*,\varepsilon))) \subseteq \tau(C_i(Z))$**.** Consider an arbitrary $\Phi \in \tau(C_i(Z))$ and an arbitrary $\Xi \notin \tau(C_i(Z))$. For ε small enough, we have that

$$\hat{u}(\Phi; r(Z;Y^*,\varepsilon)) - \hat{u}(\Xi; r(Z;Y^*,\varepsilon)); Y^*,\varepsilon) \geq \hat{u}(\Phi;Z) - \hat{u}(\Xi;Z) - |\Phi \ominus \Xi|\varepsilon$$
$$> 0,$$

where the first inequality follows from the definition of r and the second inequality follows as Φ is associated with an optimal set of contracts at Z, Ξ is not associated with an optimal set of contracts at Z, and ε is sufficiently small. Thus, $\Xi \notin \tau(C_i(r(Z;Y^*,\varepsilon)))$ and so $\tau(C_i(r(Z;Y^*,\varepsilon))) \subseteq \tau(C_i(Z))$.

Fact 4: $\tau(C_i(r(r(Z;Y^*,\varepsilon));\hat{Z}^*,\delta))) \subseteq \tau(C_i(r(Z;Y^*,\varepsilon)))$**.** Consider an arbitrary $\Phi \in \tau(C_i(r(Z;Y^*,\varepsilon)))$ and an arbitrary $\Xi \notin \tau(C_i(r(Z;Y^*,\varepsilon)))$. For δ small enough, we have that

$$\hat{u}(\Phi; r(r(Z;Y^*,\varepsilon);\hat{Z}^*,\delta)) - \hat{u}(\Xi; r(r(Z;Y^*,\varepsilon);\hat{Z}^*,\delta))$$
$$\geq \hat{u}(\Phi; r(Z;Y^*,\varepsilon)) - \hat{u}(\Xi; r(Z;Y^*,\varepsilon)) - |\Phi \ominus \Xi|\delta$$
$$> 0,$$

where the first inequality follows from the definition of r and the second inequality follows as Φ is associated with an optimal set of contracts at $r(Z;Y^*,\varepsilon)$, Ξ is not associated with an optimal set of contracts at $r(Z;Y^*,\varepsilon)$, and δ is sufficiently small. Thus, $\Xi \notin \tau(C_i(r(r(Z;Y^*,\varepsilon);\hat{Z}^*,\delta)))$ and so we have that $\tau(C_i(r(r(Z;Y^*,\varepsilon);\hat{Z}^*,\delta))) \subseteq \tau(C_i(r(Z;Y^*,\varepsilon)))$.

Fact 5: $C_i(r(r(Z;Y^*,\varepsilon);\hat{Z}^*,\delta)) = \{r(\hat{Z}^*;\hat{Z}^*,\delta)\}$**.** We have that for any feasible $W \subseteq$

$r(Z;Y^*,\varepsilon)$ such that $W \neq \hat{Z}^*$,[14]

$$\begin{aligned} U_i(r(\hat{Z}^*;\hat{Z}^*,\delta)) - U_i(r(W;\hat{Z}^*,\delta)) &= U_i(\hat{Z}^*) - U_i(W) + |\hat{Z}^* \ominus W|\delta \\ &\geq |\hat{Z}^* \ominus W|\delta \\ &> 0 \end{aligned}$$

where the equality follows from the definition of r, the first inequality follows from the fact that $\hat{Z}^*$ is optimal at $r(Z;Y^*,\varepsilon)$, i.e., $\hat{Z}^* \in C_i(r(Z;Y^*,\varepsilon))$, and the last inequality follows as $W \neq \hat{Z}^*$. Thus $C_i(r(r(Z;Y^*,\varepsilon);\hat{Z}^*,\delta)) = \{r(\hat{Z}^*;\hat{Z}^*,\delta)\}$.

Combining Facts 1 and 2 shows that there is a unique element of $\tau(C_i(r(r(Y;Y^*,\varepsilon);\hat{Z}^*,\delta)))$ and, since $r(r(Y;Y^*,\varepsilon);\hat{Z}^*,\delta)$ is a finite set, there must therefore exist a unique

$$\tilde{Y}^* \in C_i(r(r(Y;Y^*,\varepsilon);\hat{Z}^*,\delta)).$$

Fact 5 shows that $\tilde{Z}^* \equiv r(\hat{Z}^*;\hat{Z}^*,\delta)$ is the unique element of $C_i(r(r(Z;Y^*,\varepsilon);\hat{Z}^*,\delta))$. Thus, as $[r(r(Y;Y^*,\varepsilon);\hat{Z}^*,\delta)]_{\to i} \subseteq [r(r(Z;Y^*,\varepsilon);\hat{Z}^*,\delta)]_{\to i}$ by Observation 1 (as $Y_{\to i} \subseteq Z_{\to i}$) and $[r(r(Y;Y^*,\varepsilon);\hat{Z}^*,\delta)]_{i\to} = [r(r(Z;Y^*,\varepsilon);\hat{Z}^*,\delta)]_{i\to}$ (as $Y_{i\to} = Z_{i\to}$), Step 1 of the proof implies that

$$|\tilde{Z}^*_{\to i}| - |\tilde{Z}^*_{i\to}| \geq |\tilde{Y}^*_{\to i}| - |\tilde{Y}^*_{i\to}| \tag{17a}$$

$$[r(r(Y;Y^*,\varepsilon);\hat{Z}^*,\delta)]_{\to i} \smallsetminus \tilde{Y}^*_{\to i} \subseteq [r(r(Z;Y^*,\varepsilon);\hat{Z}^*,\delta)]_{\to i} \smallsetminus \tilde{Z}^*_{\to i} \tag{17b}$$

$$\tilde{Y}^*_{i\to} \subseteq \tilde{Z}^*_{i\to}. \tag{17c}$$

Each contract (ω, p_ω) in $\tilde{Y}^*_{\to i}$ has the property that p_ω is the minimal price associated with ω among all prices associated with ω by some contract in $r(r(Y;Y^*,\varepsilon);\hat{Z}^*,\delta)$ as $\tilde{Y}^*$ is optimal at $r(r(Y;Y^*,\varepsilon);\hat{Z}^*,\delta)$. Similarly, each contract (ω, p_ω) in $\tilde{Z}^*_{\to i}$ has the

[14] Note that there is a natural one-to-one correspondence between (feasible) subsets of Z, (feasible) subsets of $r(Z;Y^*,\varepsilon)$, and (feasible) subsets of $r(r(Z;Y^*,\varepsilon);\hat{Z}^*,\delta)$.

property that p_ω is the minimal price associated with ω among all prices associated with ω by some contract in $r(r(Z; Y^*, \varepsilon); \hat{Z}^*, \delta)$ as $\tilde{Z}^*$ is optimal at $r(r(Z; Y^*, \varepsilon); \hat{Z}^*, \delta)$. Moreover, each contract $(\omega, p_\omega) \in \tilde{Y}^*_{i\to}$ has the property that p_ω is the maximal price associated with ω among all contracts associated with ω in $r(r(Y; Y^*, \varepsilon); \hat{Z}^*, \delta)$, as $\tilde{Y}^*$ is optimal at $r(r(Y; Y^*, \varepsilon); \hat{Z}^*, \delta)$. Similarly, each contract $(\omega, p_\omega) \in \tilde{Z}^*_{i\to}$ has the property that p_ω is the maximal price associated with ω among all contracts associated with ω in $r(r(Z; Y^*, \varepsilon); \hat{Z}^*, \delta)$, as $\tilde{Z}^*$ is optimal at $r(r(Z; Y^*, \varepsilon); \hat{Z}^*, \delta)$. We thus rewrite (17b) and (17c) (while maintaining (17a)) as

$$|\tilde{Z}^*_{\to i}| - |\tilde{Z}^*_{i\to}| \geq |\tilde{Y}^*_{\to i}| - |\tilde{Y}^*_{i\to}| \tag{18a}$$

$$\left[\left\{\begin{array}{c} (\omega, p_\omega) \in r(r(Y; Y^*, \varepsilon); \hat{Z}^*, \delta) : \\ \omega \notin \tau(\tilde{Y}^*_{\to i}) \text{ or} \\ \exists (\omega, \bar{p}_\omega) \in r(r(Y; Y^*, \varepsilon); \hat{Z}^*, \delta) \\ \text{such that } \bar{p}_\omega < p_\omega \end{array}\right\}\right]_{\to i} \subseteq \left[\left\{\begin{array}{c} (\omega, p_\omega) \in r(r(Z; Y^*, \varepsilon); \hat{Z}^*, \delta) : \\ \omega \notin \tau(\tilde{Z}^*_{\to i}) \text{ or} \\ \exists (\omega, \bar{p}_\omega) \in r(r(Z; Y^*, \varepsilon); \hat{Z}^*, \delta) \\ \text{such that } \bar{p}_\omega < p_\omega \end{array}\right\}\right]_{\to i} \tag{18b}$$

$$\left[\left\{\begin{array}{c} (\omega, p_\omega) \in r(r(Y; Y^*, \varepsilon); \hat{Z}^*, \delta) : \\ \omega \in \tau(\tilde{Y}^*_{\to i}) \text{ and} \\ \nexists (\omega, \bar{p}_\omega) \in r(r(Y; Y^*, \varepsilon); \hat{Z}^*, \delta) \\ \text{such that } \bar{p}_\omega < p_\omega \end{array}\right\}\right]_{\to i} \subseteq \left[\left\{\begin{array}{c} (\omega, p_\omega) \in r(r(Z; Y^*, \varepsilon); \hat{Z}^*, \delta) : \\ \omega \in \tau(\tilde{Z}^*_{i\to}) \text{ and} \\ \nexists (\omega, \bar{p}_\omega) \in r(r(Z; Y^*, \varepsilon); \hat{Z}^*, \delta) \\ \text{such that } \bar{p}_\omega > p_\omega \end{array}\right\}\right]_{i\to}. \tag{18c}$$

Combining Facts 1 and 2 yields that $\tau(Y^*) = \tau(\tilde{Y}^*)$, implying that $|Y^*_{\to i}| = |\tilde{Y}^*_{\to i}|$ and

$|Y^*_{i\to}| = |\tilde{Y}^*_{i\to}|$, and so we have

$$|\tilde{Z}^*_{\to i}| - |\tilde{Z}^*_{i\to}| \geq |Y^*_{\to i}| - |Y^*_{i\to}| \tag{19a}$$

$$\left[\left\{\begin{array}{c} (\omega, p_\omega) \in r(r(Y; Y^*, \varepsilon); \hat{Z}^*, \delta) : \\ \omega \notin \tau(Y^*_{\to i}) \text{ or} \\ \exists(\omega, \bar{p}_\omega) \in r(r(Y; Y^*, \varepsilon); \hat{Z}^*, \delta) \\ \text{such that } \bar{p}_\omega < p_\omega \end{array}\right\}\right]_{\to i} \subseteq \left[\left\{\begin{array}{c} (\omega, p_\omega) \in r(r(Z; Y^*, \varepsilon); \hat{Z}^*, \delta) : \\ \omega \notin \tau(\tilde{Z}^*_{\to i}) \text{ or} \\ \exists(\omega, \bar{p}_\omega) \in r(r(Z; Y^*, \varepsilon); \hat{Z}^*, \delta) \\ \text{such that } \bar{p}_\omega < p_\omega \end{array}\right\}\right]_{\to i} \tag{19b}$$

$$\left[\left\{\begin{array}{c} (\omega, p_\omega) \in r(r(Y; Y^*, \varepsilon); \hat{Z}^*, \delta) : \\ \omega \in \tau(Y^*_{i\to}) \text{ and} \\ \nexists(\omega, \bar{p}_\omega) \in r(r(Y; Y^*, \varepsilon); \hat{Z}^*, \delta) \\ \text{such that } \bar{p}_\omega > p_\omega \end{array}\right\}\right]_{i\to} \subseteq \left[\left\{\begin{array}{c} (\omega, p_\omega) \in r(r(Z; Y^*, \varepsilon); \hat{Z}^*, \delta) : \\ \omega \in \tau(\tilde{Z}^*_{i\to}) \text{ and} \\ \nexists(\omega, \bar{p}_\omega) \in r(r(Z; Y^*, \varepsilon); \hat{Z}^*, \delta) \\ \text{such that } \bar{p}_\omega > p_\omega \end{array}\right\}\right]_{i\to}. \tag{19c}$$

Similarly, combining Facts 3–5 yields that there exists $Z^* \in C_i(Z)$ such that $\tau(Z^*) = \tau(\tilde{Z}^*)$, implying $|Z^*_{\to i}| = |\tilde{Z}^*_{\to i}|$ and $|Z^*_{i\to}| = |\tilde{Z}^*_{i\to}|$, and so we have

$$|Z^*_{\to i}| - |Z^*_{i\to}| \geq |Y^*_{\to i}| - |Y^*_{i\to}| \tag{20a}$$

$$\left[\left\{\begin{array}{c} (\omega, p_\omega) \in r(r(Y; Y^*, \varepsilon); \hat{Z}^*, \delta) : \\ \omega \notin \tau(Y^*_{\to i}) \text{ or} \\ \exists(\omega, \bar{p}_\omega) \in r(r(Y; Y^*, \varepsilon); \hat{Z}^*, \delta) \\ \text{such that } \bar{p}_\omega < p_\omega \end{array}\right\}\right]_{\to i} \subseteq \left[\left\{\begin{array}{c} (\omega, p_\omega) \in r(r(Z; Y^*, \varepsilon); \hat{Z}^*, \delta) : \\ \omega \notin \tau(Z^*_{\to i}) \text{ or} \\ \exists(\omega, \bar{p}_\omega) \in r(r(Z; Y^*, \varepsilon); \hat{Z}^*, \delta) \\ \text{such that } \bar{p}_\omega < p_\omega \end{array}\right\}\right]_{\to i} \tag{20b}$$

$$\left[\left\{\begin{array}{c} (\omega, p_\omega) \in r(r(Y; Y^*, \varepsilon); \hat{Z}^*, \delta) : \\ \omega \in \tau(Y^*_{i\to}) \text{ and} \\ \nexists(\omega, \bar{p}_\omega) \in r(r(Y; Y^*, \varepsilon); \hat{Z}^*, \delta) \\ \text{such that } \bar{p}_\omega > p_\omega \end{array}\right\}\right]_{i\to} \subseteq \left[\left\{\begin{array}{c} (\omega, p_\omega) \in r(r(Z; Y^*, \varepsilon); \hat{Z}^*, \delta) : \\ \omega \in \tau(Z^*_{i\to}) \text{ and} \\ \nexists(\omega, \bar{p}_\omega) \in r(r(Z; Y^*, \varepsilon); \hat{Z}^*, \delta) \\ \text{such that } \bar{p}_\omega > p_\omega \end{array}\right\}\right]_{i\to}. \tag{20c}$$

We have, by (20c) that, if $\omega \in \tau(Y^*_{i\to})$, then $\omega \in \tau(Z^*_{i\to})$; moreover, since $Y_{i\to} = Z_{i\to}$ by assumption, the set of prices corresponding to a given $\omega \in \Omega_{i\to}$ is the same in Y and Z.

We thus rewrite (20c) (while maintaining (20a) and (20b)) as

$$|Z^*_{\to i}| - |Z^*_{i\to}| \geq |Y^*_{\to i}| - |Y^*_{i\to}| \tag{21a}$$

$$\left[\left\{\begin{array}{c}(\omega, p_\omega) \in r(r(Y; Y^*, \varepsilon); \hat{Z}^*, \delta) : \\ \omega \notin \tau(Y^*_{\to i}) \text{ or} \\ \exists(\omega, \bar{p}_\omega) \in r(r(Y; Y^*, \varepsilon); \hat{Z}^*, \delta) \\ \text{such that } \bar{p}_\omega < p_\omega\end{array}\right\}\right]_{\to i} \subseteq \left[\left\{\begin{array}{c}(\omega, p_\omega) \in r(r(Z; Y^*, \varepsilon); \hat{Z}^*, \delta) : \\ \omega \notin \tau(Z^*_{\to i}) \text{ or} \\ \exists(\omega, \bar{p}_\omega) \in r(r(Z; Y^*, \varepsilon); \hat{Z}^*, \delta) \\ \text{such that } \bar{p}_\omega < p_\omega\end{array}\right\}\right]_{\to i} \tag{21b}$$

$$\left[\left\{\begin{array}{c}(\omega, p_\omega) \in Y : \\ \omega \in \tau(Y^*_{i\to}) \text{ and} \\ \nexists(\omega, \bar{p}_\omega) \in Y \\ \text{such that } \bar{p}_\omega > p_\omega\end{array}\right\}\right]_{i\to} \subseteq \left[\left\{\begin{array}{c}(\omega, p_\omega) \in Z : \\ \omega \in \tau(Z^*_{i\to}) \text{ and} \\ \nexists(\omega, \bar{p}_\omega) \in Z \\ \text{such that } \bar{p}_\omega > p_\omega\end{array}\right\}\right]_{i\to}. \tag{21c}$$

We have, by (21b) that, if $\omega \notin \tau(Y^*_{\to i})$, then $\omega \notin \tau(Z^*_{\to i})$; moreover, since $Y_{\to i} \subseteq Z_{\to i}$ by assumption, the set of prices available for a given $\omega \in \Omega_{\to i}$ is larger in Y than in Z. We thus rewrite (21b) (while maintaining (21a) and (21c)) as

$$|Z^*_{\to i}| - |Z^*_{i\to}| \geq |Y^*_{\to i}| - |Y^*_{i\to}| \tag{22a}$$

$$\left[\left\{\begin{array}{c}(\omega, p_\omega) \in Y : \\ \omega \notin \tau(Y^*_{\to i}) \text{ or} \\ \exists(\omega, \bar{p}_\omega) \in Y \\ \text{such that } \bar{p}_\omega < p_\omega\end{array}\right\}\right]_{\to i} \subseteq \left[\left\{\begin{array}{c}(\omega, p_\omega) \in Z : \\ \omega \notin \tau(Z^*_{\to i}) \text{ or} \\ \exists(\omega, \bar{p}_\omega) \in Z \\ \text{such that } \bar{p}_\omega < p_\omega\end{array}\right\}\right]_{\to i} \tag{22b}$$

$$\left[\left\{\begin{array}{c}(\omega, p_\omega) \in Y : \\ \omega \in \tau(Y^*_{i\to}) \text{ and} \\ \nexists(\omega, \bar{p}_\omega) \in Y \\ \text{such that } \bar{p}_\omega > p_\omega\end{array}\right\}\right]_{i\to} \subseteq \left[\left\{\begin{array}{c}(\omega, p_\omega) \in Z : \\ \omega \in \tau(Z^*_{i\to}) \text{ and} \\ \nexists(\omega, \bar{p}_\omega) \in Z \\ \text{such that } \bar{p}_\omega > p_\omega\end{array}\right\}\right]_{i\to}. \tag{22c}$$

We rewrite this expression as

$$|Z^*_{\to i}| - |Z^*_{i\to}| \geq |Y^*_{\to i}| - |Y^*_{i\to}|$$
$$[Y \smallsetminus Y^*]_{\to i} \subseteq [Z \smallsetminus Z^*]_{\to i}$$
$$[Y^*]_{i\to} \subseteq [Z^*]_{i\to}.$$

Thus the preferences of i satisfy the requirements of Part 1 of Definition 11.

The proof that the preferences of i satisfy the requirements of Part 2 is analogous. Combining these results, we obtain that full substitutability implies monotone–substitutability.